Online Appendix to
"Self-Fulfilling Debt Dilution"

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Appendix A Closed-Form Expressions and Derivations

In this appendix, we provide closed-form expressions for the solutions to the planning problem as well as equilibrium objects. We also include some notes on the underlying derivations.

A.1 The Efficient Borrowing Allocation

The conjectured policy function for consumption in the borrowing allocation is given in (5). As $\underline{V}$ is a stationary point, we immediately have:

$$P_B^\star(\underline{V}) = \frac{y - C^\star(\underline{V})}{r+\lambda}.$$

Given this boundary condition and the consumption policy function, we solve the ODE (P) to obtain:

(i) For $v \in [\underline{V}, \overline{V}]$:

$$P_B^\star(v) \equiv \frac{1}{r+\lambda}\left[y - \overline{C} + \frac{(\overline{C} + \lambda\overline{V} - (\rho+\lambda)v)^{\frac{r+\lambda}{\rho+\lambda}}}{(\overline{C} + \lambda\overline{V} - (\rho+\lambda)\underline{V})^{\frac{r-\rho}{\rho+\lambda}}}\right].$$

(ii) For $v \in (\overline{V}, V_{max}]$:

$$P_B^\star(v) \equiv \frac{1}{r}\left[y - \overline{C} + (\overline{C} - y + rP_B^\star(\overline{V}))\frac{(\overline{C} - \rho v)^{\frac{r}{\rho}}}{(\overline{C} - \rho\overline{V})^{\frac{r}{\rho}}}\right].$$

A.2 Efficient Saving

For the efficient saving allocation, the conjecture is that the Safe Zone is an absorbing state. In particular, $\overline{V}$ is a stationary point, which pins down $P_S^\star(\overline{V}) = (y - \rho\overline{V})/r$. Given this boundary condition and the policy function (9), we solve (P) for the Safe Zone to obtain:

$$P_S^\star(v) \equiv \frac{1}{r}\left[y - \overline{C} + \left(\overline{C} - \rho\overline{V}\right)^{\frac{\rho-r}{\rho}}\left(\overline{C} - \rho v\right)^{\frac{r}{\rho}}\right] \text{ for } v \in [\overline{V}, V_{max}]. \quad (1)$$

For the saving region of the Crisis Zone, consumption is at its lower bound, $\underline{C}$. Solving (P) for $v \in [\underline{V}, \overline{V})$, using $P_S^\star(\overline{V})$ from above as a boundary condition, we obtain the planner's value under saving:

$$\hat{P}(v) \equiv \frac{1}{r+\lambda}\left[y - \underline{C} + (\underline{C} - y + (r+\lambda)P_S^\star(\overline{V}))\left(\frac{\underline{C} + \lambda\overline{V} - (\rho+\lambda)v}{\underline{C} - \rho\overline{V}}\right)^{\frac{r+\lambda}{\rho+\lambda}}\right]. \tag{2}$$

Per equation (12), $P_S^\star(v)$ in the Crisis Zone is the maximum of $\hat{P}$ and $P_B^\star$. Straightforward differentiation indicate that $\hat{P}$ and $P_B^\star$ cross at most once.

A.3 The Borrowing Equilibrium

In the Crisis Zone $(\underline{b}_B, \overline{b}_B]$, $q_B(b) = \underline{q}$. Turning to the Safe Zone, recall that the conjectured consumption policy function in the borrowing equilibrium is the same as the planner's borrowing policy, (5). With this policy and the boundary $q_B(\overline{b}_B) = \underline{q}$, the solution to (20) is defined implicitly by:

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}}\right)^{\frac{r}{r+\delta}} = \frac{\overline{C} - y + rq_B(b)b}{\overline{C} - y + r\underline{q}\underline{b}_B}. \tag{3}$$

For each $b \in [0, \underline{b}_B)$, there is a unique solution for $q_B(b) \in [\underline{q}, 1]$. Recall that for $b < 0$, we have $q_B(b) = 1$ regardless of the government's policies.[1]

The government's value in the borrowing equilibrium is obtained by inverting $P_B^\star$. Specifically:

$$V_B(b) = \begin{cases} \frac{1}{\rho}\left(\overline{C} - (\overline{C} - \rho\overline{V})\left(\frac{\overline{C} - y + rq_B(b)b}{\overline{C} - y + r\underline{q}\underline{b}_B}\right)^{\frac{\rho}{r}}\right) & \text{for } b \in [-\overline{a}, \underline{b}_B] \\ \frac{1}{\rho+\lambda}\left(\overline{C} + \lambda\overline{V} - \frac{\left(\overline{C} - y + (r+\lambda)\underline{q}b\right)^{\frac{\rho+\lambda}{r+\lambda}}}{\left(\overline{C} - y + (r+\lambda)\underline{q}\overline{b}_B\right)^{\frac{\rho-r}{r+\lambda}}}\right) & \text{for } b \in (\underline{b}_B, \overline{b}_B], \end{cases} \tag{4}$$

where $\overline{a} \equiv (\overline{C} - y)/r$ is the maximal net inflows that can be consumed by the government.

A.4 The Saving Equilibrium

The saving equilibrium objects in the Safe Zone is straightforward: because it is an absorbing region, there is no risk of default starting from $b \le \underline{b}_S$. Hence, the price is one, $q_S(b) = 1$ and the values and consumption are equivalent to their efficient counterparts. That is, inverting $P_S^\star$ we

[1] Note that there may be a discontinuity in q_B at $b = 0$. Recall that at points of discontinuity, we impose that debt buybacks occur at a price of one in the neighborhood around a discontinuity. This restriction eliminates the technical complication of the government attempting to issue debt at one price and near-simultaneously repurchasing at a lower price in an attempt to exploit this discontinuity. The restriction we impose ensures that the choice set is convex despite the discontinuity in price, and hence the government has no motive to "mix" by moving consumption back and forth while keeping debt at the point of discontinuity.

obtain:

$$V_S(b) = \rho^{-1}\left(\overline{C} - (\overline{C} - \rho\overline{V})\left(\frac{\overline{C} - y + rb}{\overline{C} - y + r\underline{b}_S}\right)^{\frac{\rho}{r}}\right) \text{ for } b \in [-\overline{a}, \underline{b}_S] \tag{5}$$

where $\underline{b}_S \equiv (y - \rho\overline{V})/r$. The consumption policy is $\overline{C}$ for $b < \underline{b}_S$, and $\rho\overline{V}$ at $\underline{b}_S$.

Turning to the Crisis Zone, we begin with the saving region. Let $\{\hat{V}, \hat{C}, \hat{q}\}$ denote the conjectured equilibrium objects in the saving region of the Crisis Zone. In the saving region, we have to deviate from the prescription of the efficient allocation. The reason is that the efficient savings policy, which sets consumption at its lower bound $\underline{C}$, cannot be sustained in a competitive equilibrium. That is, the efficient savings rate is not privately optimal in an equilibrium with long-term bonds.

We conjecture instead that the government saves by consuming at an interior optimum.[2] When consumption is interior, the linearity of the government's objective function in (17) implies that it is indifferent across alternative consumption choices, including the consumption level that sets $\dot{b} = 0$. Hence, the government must be indifferent between the equilibrium consumption strategy and its associated stationary value:[3]

$$\hat{V}(b) \equiv \frac{y - [r + \delta(1 - \hat{q}(b))]b + \lambda\overline{V}}{\rho + \lambda}. \tag{6}$$

From the first-order condition in (17), interior consumption requires $\hat{V}'(b) = -\hat{q}(b)$. Using this, differentiating (6), and solving the resulting ODE with $\hat{q}(\underline{b}_S) = 1$ as a boundary condition yields

$$\hat{q}(b) \equiv \frac{r + \delta + \left(\frac{b}{\underline{b}_S}\right)^{-\frac{\rho+\lambda+\delta}{\delta}}(\lambda + \rho - r)}{\rho + \lambda + \delta}. \tag{7}$$

The lenders' break-even condition (19) requires $\hat{q}'(b)\dot{b} = (r + \delta + \lambda)\hat{q}(b) - (r + \delta)$. Hence, we can solve for the conjectured debt dynamics:

$$\dot{b} = -\delta b\left(\frac{\hat{q}(b) - \underline{q}}{\hat{q}(b) - \underline{q} + \frac{(\rho - r)\hat{q}(b)}{r + \delta + \lambda}}\right) \equiv f(b). \tag{8}$$

Using (16), we obtain the associated consumption:

$$\hat{C}(b) \equiv y - [r + \delta(1 - \hat{q}(b))]b + \hat{q}(b)f(b). \tag{9}$$

[2] Throughout the following analysis, we assume $\underline{C}$ is sufficiently low that an interior consumption choice is feasible.

[3] The fact that the government's value is equal to the stationary value while consumption is interior is discussed in Tourre (2017) and DeMarzo, He and Tourre (2018). The authors give an interpretation of a durable monopolist in the spirit of the Coase conjecture.

The borrowing region of the Crisis Zone is also an absorbing state and corresponds to the equilibrium discussed in the previous subsection. Note that in this region, the price is $\underline{q}$. In the Crisis Zone, $V_S(b) = \max\langle \hat{V}(b), V_B(b)\rangle$. As before, b^I is the intersection point of these two alternatives. If no such $b^I \in [\underline{b}_S, \overline{b}_B]$ exists, we set it to $\overline{b}_S$. The value of $\overline{b}_S$ is such that $V_S(\overline{b}_S) = \underline{V}$, and we define $\boldsymbol{B}_S \equiv [-\overline{a}, \overline{b}_S]$.[4]

The saving equilibrium value in the Crisis Zone is therefore:

$$V_S(b) \equiv \begin{cases} \hat{V}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ V_B(b) & \text{for } b \in (b^I, \overline{b}_S]; \end{cases} \tag{10}$$

and the consumption policy is

$$C_S(b) \equiv \begin{cases} \hat{C}(b) & \text{for } b \in (\underline{b}_S, b^I] \\ C_B(b) & \text{for } b \in (b^I, \overline{b}_S]. \end{cases} \tag{11}$$

The equilibrium price schedule is:

$$q_S(b) \equiv \begin{cases} 1 & \text{for } b \in [-\overline{a}, \underline{b}_S] \\ \hat{q}(b) & \text{for } b \in [\underline{b}_S, b^I] \\ \underline{q} & \text{for } b \in (b^I, \overline{b}_S]; \end{cases} \tag{12}$$

Appendix B Additional Results

In this appendix, we state state four results. The first two allows us to provide a characterization of a solution to the planner's problem. The next two are the same results for the government's problem in a competitive equilibrium.

The value function $P^\star(v)$ has the following standard properties:

Lemma B.1. *The solution to the planner's problem, $P^\star(v)$, is bounded and Lipschitz continuous.*

Proof. The proof is in Appendix C. □

Lemma B.1 states that $P^\star$ is bounded and Lipschitz continuous, and hence differentiable almost everywhere. However, there may be isolated points of non-differentiability. At such points, $P^\star$ satisfies (P) in the viscosity sense. In particular:

Proposition B.1. *Suppose a bounded, Lipschitz continuous function $p(v)$ with domain $\mathbb{V}$ has the following properties:*

(i) p satisfies (P) *at all points of differentiability;*

[4] If $b^I < \overline{b}_B$, then q_S is discontinuous at b^I, which is the case depicted in Figure 3. As previously discussed when stating the government's problem, and echoed in footnote 1, we rule out the government issuing at $q_S(b^I)$ and then immediately repurchasing at $\lim_{b'\downarrow b^I} q_S(b') < q_S(b^I)$ in an attempt to set $\dot{b} = 0$ by alternating between issuing and repurchasing. Let us also note that the multiplicity result we obtain later on does not hinge on this particular issue: it is possible to obtain parameter values such that $b^I = \overline{b}_S$ and for which multiple equilibria coexist.

(ii) *If* $\lim_{v\uparrow\overline{V}} p'(v) > \lim_{v\downarrow\overline{V}} p'(v)$ *and* $\lim_{v\uparrow\overline{V}} p'(v) \geq -1$*, then* $p(\overline{V}) = (y - \rho\overline{V})/r$*;*

(iii) *At a point of non-differentiability* $\tilde{v} \neq \overline{V}$*, we have* $\lim_{v\uparrow\tilde{v}} p'(v) < \lim_{v\downarrow\tilde{v}} p'(v)$*;*

(iv) *If* $p'(\underline{V}) < -1$*, then* $p(\underline{V}) = (y - \rho\underline{V} + \lambda(\overline{V} - \underline{V}))/(r + \lambda)$*;*[5] *and*

(v) $p'(V_{max}) \leq -1$*;*

then $p(v) = P^{\star}(v)$*.*

Proof. The proof is in Appendix E. □

The first condition of the proposition ensures that the candidate value function satisfies the HJB wherever it is smooth. The second condition concerns the case when $\overline{V}$ is a locally stable stationary point; this will be relevant when we consider an efficient "saving allocation" defined below. The third condition states that any other point of non-differentiability has a "convex" kink. The final two conditions are sufficient to ensure that v remains in $\mathbb{V}$.

The counterpart to Lemma B.1 for the equilibrium value function is:

Lemma B.2. *In any competitive equilibrium such that* $q(b) \in [\underline{q}, 1]$ *for* $b \in \boldsymbol{B} = [-\overline{a}, \overline{b}]$*,* V *is bounded, strictly decreasing, and Lipschitz continuous on* $\boldsymbol{B}$*.*

Proof. The proof is in Appendix C. □

The counterpart to Proposition B.1 for the government's equilibrium problem (17) is:

Proposition B.2. *Consider the government's problem given a compact debt domain* $\boldsymbol{B}$ *and a price schedule* $q : \boldsymbol{B} \to [\underline{q}, 1]$ *that has a (bounded) derivative at almost all points in* $\boldsymbol{B}$*. If a strictly decreasing, Lipschitz continuous function* $v : \boldsymbol{B} \to [\underline{V}, \overline{C}/\rho]$ *has the following properties:*

(i) v *satisfies* (17) *at all points of differentiability;*

(ii) *If* $\lim_{b\uparrow\underline{b}} v'(b) > \lim_{b\downarrow\underline{b}} v'(b)$*, then* $\rho v(\underline{b}) = \rho\overline{V} = y - [r + \delta(1 - q(\underline{b}))]\underline{b}$*;*

(iii) *At a point of non-differentiability* $\tilde{b} \neq \underline{b}$*, we have* $\lim_{b\uparrow\tilde{b}} v'(b) < \lim_{b\downarrow\tilde{b}} v'(b)$*;*

(iv) $\rho v(-\overline{a}) = \overline{C}$*; and*

(v) $(\rho + \lambda)v(\overline{b}) = y - [r + \delta(1 - q(\overline{b}))]\overline{b} + \lambda\overline{V}$*;*

then $v(b) = V(b)$ *is the government's value function.*

Proof. The proof is in Appendix E. □

The conditions listed in the proposition are similar to those from Proposition B.1. Namely, that the value function satisfies the HJB equation with equality wherever smooth; there may be a local attractor that corresponds to $\underline{b}$ if the government saves; other points of non-differentiability have convex kinks; and the endpoints of the domain deliver the value of holding debt constant.[6]

[5] For the endpoints of $\mathbb{V}$, we interpret $p'(\underline{V}) \equiv \lim_{v\downarrow\underline{V}} p'(v)$ and $p'(V_{max}) \equiv \lim_{v\uparrow V_{max}} p'(v)$.

[6] Condition (v), at $\overline{b}$, is stronger than necessary, as the key requirement is that $\dot{b} \leq 0$ at the upper bound on debt; however, in the equilibria described below, the stronger condition is always satisfied.

Appendix C Proofs

This appendix contains all proofs except those for Propositions B.1 and B.2, which are presented in the Online Appendix, along with a discussion of viscosity solutions more generally.

C.1 Proof of Lemma 1

Proof. To generate a contradiction, suppose there is an efficient allocation $\{\boldsymbol{c}, T\}$, with $T < \infty$. Note from (1) we have $V(T, \boldsymbol{c}) = \underline{V}$. To see this, suppose instead that $V(T, \boldsymbol{c}) > \underline{V}$; that is,

$$\begin{aligned} V(T, \boldsymbol{c}) &= \sup_{T' \geq t} \int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s) \mathrm{d}s + e^{-(\rho+\lambda)(T'-T)} \underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \mathrm{d}s \\ &> \underline{V}. \end{aligned}$$

Hence, there exists a $T' > T$ such that

$$\int_T^{T'} e^{-(\rho+\lambda)(s-T)} c(s) \mathrm{d}s + e^{-(\rho+\lambda)(T'-T)} \underline{V} + \lambda \int_T^{T'} e^{-(\rho+\lambda)(s-T)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \mathrm{d}s > \underline{V}.$$

This implies at time $t < T$,

$$\begin{aligned} &\int_t^{T} e^{-(\rho+\lambda)(s-t)} c(s) \mathrm{d}s + e^{-(\rho+\lambda)(T-t)} \underline{V} + \lambda \int_t^{T} e^{-(\rho+\lambda)(s-t)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \mathrm{d}s < \\ &\int_t^{T'} e^{-(\rho+\lambda)(s-t)} c(s) \mathrm{d}s + e^{-(\rho+\lambda)(T'-t)} \underline{V} + \lambda \int_t^{T'} e^{-(\rho+\lambda)(s-t)} \max\langle V(s, \boldsymbol{c}), \overline{V} \rangle \mathrm{d}s. \end{aligned}$$

Hence, T was never a sup of the original problem. This establishes that $V(T, \boldsymbol{c}) = \underline{V}$.

Now consider an alternative allocation $(\tilde{\boldsymbol{c}}, \infty)$. The alternative consumption allocation equals $\boldsymbol{c}$ for $t < T$, but differs for $t \geq T$. We choose $\tilde{c}(t) = (\rho + \lambda)\underline{V} - \lambda \overline{V} < y$ for $t \geq T$ so that for all $t \geq T$:

$$\begin{aligned} V(t, \tilde{\boldsymbol{c}}) &= \frac{\tilde{c}(t) + \lambda \overline{V}}{\rho + \lambda} \\ &= \frac{(\rho + \lambda)\underline{V} - \lambda \overline{V} + \lambda \overline{V}}{\rho + \lambda} \\ &= \underline{V}. \end{aligned}$$

Thus, $V(0; \boldsymbol{c}) = V(0; \tilde{\boldsymbol{c}})$. Moreover, the alternative allocation delivers strictly more than zero to the lender in expectation for $t \geq T$ as $\tilde{c}(t) < y$. As the government is indifferent and the lender receives strictly more in expected present value, the original allocation is not efficient. □

C.2 Proof of Lemma B.1

Proof. Lemma 1 allows us to set $T = \infty$ in the planning problem (3) to obtain

$$P^\star(v) = \sup_{\boldsymbol{c} \in \boldsymbol{C},} \int_0^\infty e^{-\int_0^t r + \mathbb{1}_{[v(s)<\overline{V}]}\lambda ds}[y - c(t)]\mathrm{d}t \tag{13}$$

$$\text{subject to} \begin{cases} v(0) &= v \\ \dot{v}(t) &= -c(t) + \rho v(t) - \mathbb{1}_{[v(t)<\overline{V}]}\lambda\left[\overline{V} - v(t)\right], \end{cases}$$

defined on the domain $v \in \mathbb{V}$. $P^\star$ is bounded above by $(y - \underline{C})/r$ and below by $(y - \overline{C})/r$. To see that $P^\star$ is Lipschitz continuous in v, consider $v_1, v_2 \in \mathbb{V}$, with $v_2 > v_1$. A feasible strategy starting from $v(0) = v_2$ is to set consumption to $\overline{C}$ until $v(t) = v_1$. Let Δ denote the time $v(t)$ reaches v_1. Suppose $v(t) > \overline{V}$ for $t \in [0, \Delta_1)$ and $v(t) < \overline{V}$ for $t \in (\Delta_1, \Delta]$. Let $\Delta_2 = \Delta - \Delta_1$. If $v_2 < \overline{V}$, then $\Delta_1 = 0$ and if $v_1 > \overline{V}$, then $\Delta_2 = 0$. The dynamics of $v(t)$ imply

$$\begin{aligned} e^{-\rho\Delta_1} &= \frac{\overline{C} - \rho \max\{v_2, \overline{V}\}}{\overline{C} - \rho \max\{v_1, \overline{V}\}} \\ e^{-(\rho+\lambda)\Delta_2} &= \frac{\overline{C} + \lambda\overline{V} - (\rho + \lambda)\min\{v_2, \overline{V}\}}{\overline{C} + \lambda\overline{V} - (\rho + \lambda)\min\{v_1, \overline{V}\}}. \end{aligned}$$

Using this, one can show that

$$1 - e^{-\rho\Delta_1 - (\rho+\lambda)\Delta_2} \leq L|v_2 - v_1|, \tag{14}$$

with $L \equiv (\rho + \lambda)/(\overline{C} - \rho V_{max}) \in (0, \infty)$.

As this is a feasible strategy for v_2, integrating the objective function, we obtain

$$P^\star(v_2) \geq (y - \overline{C})\left(\frac{1 - e^{-r\Delta_1}}{r} + \frac{e^{-r\Delta_1}\left(1 - e^{-(r+\lambda)\Delta_2}\right)}{r + \lambda}\right) + e^{-r\Delta_1 - (r+\lambda)\Delta_2}P^\star(v_1).$$

As $y < \overline{C}$, we have

$$P^\star(v_2) \geq (y - \overline{C})\left(\frac{1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}}{r}\right) + e^{-r\Delta_1 - (r+\lambda)\Delta_2}P^\star(v_1),$$

which implies

$$P^\star(v_1) - P^\star(v_2) \leq \left(\frac{\overline{C} - y}{r} + P^\star(v_1)\right)\left(1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}\right).$$

As $P^\star(v_1) \le (y - \underline{C})/r$, we have

$$P^\star(v_1) - P^\star(v_2) \le \left(\frac{\overline{C} - \underline{C}}{r}\right)\left(1 - e^{-r\Delta_1 - (r+\lambda)\Delta_2}\right).$$

As $\overline{C} > \underline{C}$ and $\rho \ge r$, this implies

$$\begin{aligned} P^\star(v_1) - P^\star(v_2) &\le \left(\frac{\overline{C} - \underline{C}}{r}\right)\left(1 - e^{-\rho\Delta_1 - (\rho+\lambda)\Delta_2}\right) \\ &\le \left(\frac{\overline{C} - \underline{C}}{r}\right) L|v_2 - v_1|, \end{aligned}$$

where the second line uses (14). As $v_1 < v_2$, and hence $P^\star(v_1) \ge P^\star(v_2)$ as $P^\star$ is the efficient frontier, we have

$$|P^\star(v_1) - P^\star(v_2)| \le K|v_2 - v_1|,$$

where

$$K \equiv \left(\frac{\overline{C} - \underline{C}}{r}\right) L = \left(\frac{\overline{C} - \underline{C}}{\overline{C} - \rho V_{max}}\right)\left(\frac{\rho + \lambda}{r}\right).$$

Hence, $P^\star$ is Lipschitz continuous with coefficient $K \in (0, \infty)$. □

C.3 Proof of Proposition 1

Proof. We need to check the conditions of Proposition B.1. Note that $P_B^\star$ is bounded, Lipschitz continuous, and differentiable everywhere except $\overline{V}$, where $\lim_{v\uparrow\overline{V}} P^{\star\prime}(v) < \lim_{v\downarrow\overline{V}} P^{\star\prime}(v)$. This inequality implies that condition (ii) in the proposition is irrelevant. Condition (iii) of Proposition B.1 is satisfied trivially. Condition (iv) is satisfied by construction.

At points of differentiability, the first-order condition for consumption requires $P_B^{\star\prime}(v) \le -1$ for $c = \overline{C}$ to be optimal. Starting with $v \in [\underline{V}, \overline{V})$, differentiating the candidate function yields $P_B^{\star\prime}(v) \le -1$. Hence $\overline{C}$ is optimal, and $P_B^\star$ satisfies the HJB on this domain. Turning to $v > \overline{V}$, note that $P_B^\star(v)$ is concave on this domain. Thus, if $\lim_{v\downarrow\overline{V}} P_B^{\star\prime}(v) \le -1$, then $P_B^{\star\prime}(v) \le -1$ for $v \in (\overline{V}, V_{max}]$. We have

$$\lim_{v\downarrow\overline{V}} P_B^{\star\prime}(v) = -\frac{\overline{C} - y + rP_B^\star(\overline{V})}{\overline{C} - \rho\overline{V}}.$$

This quantity is less than -1 when $rP_B^\star(\overline{V}) \ge y - \rho\overline{V}$. This is the condition stated in the proposition. This condition is necessary and sufficient for $P_B^\star$ to satisfy the HJB on $(\overline{V}, V_{max})$. Moreover, it is sufficient to ensure that condition (v) of Proposition B.1 is satisfied. □

C.4 Proof of Proposition 2

Proof. The proposed solution $P_S^\star$ is differentiable everywhere save $\overline{V}$ and v^I. At $\overline{V}$ we have $\lim_{v\uparrow\overline{V}} P_S^{\star\prime}(v) \geq -1 \geq \lim_{v\downarrow\overline{V}} P_S^{\star\prime}$. Hence, condition (ii) of Proposition B.1 is relevant and is satisfied by the candidate value function. $P_S^\star$ satisfies condition (iii) at v^I as it features a convex kink by construction. Condition (iv) is also satisfied by construction.

On the domain $v \in (\overline{V}, V_{max}]$, we have $P_S^{\star\prime}(v) \leq -1$, and hence $P_S^\star$ satisfies the HJB as well as condition (v) of Proposition B.1.

Turning to $v < \overline{V}$, we now show that $P_S^\star(\overline{V}) \geq P_B^\star(\overline{V})$ is necessary and sufficient for $P_S^\star$ to satisfy the conditions of Proposition B.1.

For sufficiency, suppose that $P_S^\star(\overline{V}) \geq P_B^\star(\overline{V})$. Let $X \equiv \{v \in [\underline{V}, \overline{V}) | P_S^\star(v) \geq P_B^\star(v)\} = [\max\{v^I, \underline{V}\}, \overline{V})$. On the domain X, $P_S^\star(v) = \hat{P}(v)$. One can show that $\hat{P}'(v) \geq -1$ if and only if $\hat{P}(v) \geq (y - (\rho + \lambda)v + \lambda\overline{V})/(r + \lambda)$. As the latter term is the value associated with setting $\dot{v} = 0$, the inequality is satisfied as $\hat{P}(v) \geq P_B^\star(v) \geq (y - (\rho + \lambda)v + \lambda\overline{V})/(r + \lambda)$. Hence $c = \underline{C}$ is optimal on X, and the HJB is satisfied. If $\hat{P}(\underline{V}) \geq P_B^\star(\underline{V})$, then $X = [\underline{V}, \overline{V})$, and hence the HJB is satisfied on the whole domain $\mathbb{V}$. If instead there exists $v^I > \underline{V}$, then the HJB is satisfied for $v < v^I$ from Proposition 1.

For necessity, suppose instead that $P_S^\star(\overline{V}) < P_B^\star(\overline{V})$. Comparison of the slopes implies that as long as $P_S^\star(v) < P_B^\star(v)$ for $v \in [\underline{V}, \overline{V})$, then $P_S^{\star\prime}(v) < P_B^{\star\prime}(v)$, and the two lines will never cross. Moreover, $P_B^{\star\prime}(v) \leq -1$, and hence $P_S^{\star\prime}(v) < -1$. This implies that $c = \underline{C}$ is strictly sub-optimal and the HJB is violated. □

C.5 Proof of Lemma B.2

Proof. The boundedness of V follows directly from $\overline{C}/\rho \geq V(b) \geq \underline{V}$ for any $b \in \boldsymbol{B}$.

To see that V is strictly decreasing, suppose $b_1 > b_2$ for $b_1, b_2 \in \boldsymbol{B}$. If $b_2 = -\overline{a} \equiv (y - \overline{C})/\rho$, then $V(b_2) = \overline{C}/\rho > V(b_1)$, where the latter inequality follows from the budget set at $b_1 > b_2$. Now consider the following policy starting from $b_2 \in (-\overline{a}, b_1)$: Set $c = \overline{C}$ until $b(t) = b_1$. As

$$\dot{b}(t) = \frac{c + (r + \delta)b(t) - y}{q(b(t))} - \delta b,$$

and $\overline{C} > y - rb \geq y - [r + \delta(1 - q(b))]b$ for $b \geq b_2$, we have $\dot{b}(t) > 0$. Let $\tilde{t} \in (0, \infty)$ denote when $b(t) = b_1$. As it is feasible for the government to follow this policy and not default while doing so, we have

$$V(b_2) \geq \int_0^{\tilde{t}} \overline{C} dt + e^{-\rho\tilde{t}} V(b_1) = \left(1 - e^{\rho\tilde{t}}\right) \frac{\overline{C}}{\rho} + e^{-\rho\tilde{t}} V(b_1).$$

Subtracting $V(b_1)$ from both sides yields:

$$V(b_2) - V(b_1) \geq \left(1 - e^{\rho\tilde{t}}\right) \left(\frac{\overline{C}}{\rho} - V(b_1)\right) > 0.$$

For continuity, we proceed in a similar fashion. Starting from b_1, consider the policy of setting

$c = \underline{C}$ until $b(t) = b_2$. Let t^* denote the time where $b(t) = b_2$. Given that $\underline{C} < y - (r+\delta)\overline{b} \leq y - (r+\delta)b(t)$ and $q(b(t)) \in [\underline{q}, 1]$, $t^* < \infty$. Moreover, the same statements imply that

$$b_2 - b_1 \geq \int_0^{t^*} \left(\underline{C} + rb(t) - y\right) dt \geq \int_0^{t^*} \left(\underline{C} + r\overline{b} - y\right) dt = \left(\underline{C} + r\overline{b} - y\right) t^*,$$

where the first inequality follows from $q(b) \leq 1$.

The above implies that $t^* \geq L|b_1 - b_2|$, with $L \equiv \left(y - r\overline{b} - \underline{C}\right)^{-1} \in (0, \infty)$.

As this is a feasible strategy, we have

$$V(b_1) \geq \int_0^{t^*} e^{-\rho t}\underline{C}dt + e^{-\rho t^*}V(b_2) = (1 - e^{-\rho t^*})\frac{\underline{C}}{\rho} + e^{-\rho t^*}V(b_2),$$

where the inequality in the first line also reflects that the right-hand side is the value assuming the government never defaults, which is weakly below the optimal default policy. Subtracting $V(b_2)$ from both sides and rearranging, we have

$$V(b_2) - V(b_1) \leq (1 - e^{-\rho t^*})\left(V(b_2) - \frac{\underline{C}}{\rho}\right).$$

Using the fact that $\overline{C}/\rho > V(b) \geq \underline{V} > \underline{C}/\rho$ and $1 - e^{-\rho t^*} \leq t^*$, we have

$$0 < V(b_2) - V(b_1) \leq t^* \left(V(b_2) - \frac{\underline{C}}{\rho}\right) \leq L\left(\frac{\overline{C} - \underline{C}}{\rho}\right)|b_1 - b_2|.$$

Hence, $|V(b_2) - V(b_1)| \leq K|b_2 - b_1|$ with $K \equiv L\left(\overline{C} - \underline{C}\right)/\rho \in (0, \infty)$. □

C.6 Proof of Proposition 3

Proof. By construction, the price schedule q_B is consistent with the lenders' break-even condition, given the conjectured government policy. The remaining step is to verify if and when the government's policy is optimal given the conjectured q_B. Hence, to prove the proposition, we need to establish that V_B satisfies the conditions of Proposition B.2 if and only if (22) holds.

For $\overline{C}$ to be optimal for all $b < \overline{b}_B$, the first-order condition for the HJB requires $1 + V_B'(b)/q_B(b) \geq 0$ wherever $V_B'(b)$ exists. Thus, if $V_B'(b) \geq -q_B(b)$, then $c = \overline{C}$ is optimal. Recalling that V_B was constructed by assuming that the Hamiltonian is maximized at $c = \overline{C}$, then $V_B'(b) \geq -q_B(b)$ is both necessary and sufficient to verify that the HJB is satisfied at points of differentiability.

We proceed to show that (22) is equivalent to $V_B'(b) \geq -q_B(b)$ at points of differentiability.

For $b < 0$, we have

$$\rho V_B(b) = \overline{C} - (\overline{C} - \rho V_B(0))\left(\frac{\overline{C} + rb - y}{\overline{C} - y}\right)^{\frac{\rho}{r}}.$$

Note that V_B is concave on this domain. For $\overline{C}$ to be optimal, it is therefore sufficient that

$\lim_{b\uparrow 0} V_B'(b) \geq -1$. This will be true if and only if $\rho V_B(0) \geq y$. Hence, the condition in equation (22) evaluated at $b = 0$ is necessary and sufficient for the HJB to hold for $b \in (-\overline{a}, 0)$. For $b = -\overline{a} = (y - \overline{C})/\rho$, we have $V_B(-\overline{a}) = \overline{C}/\rho$, which is condition (iv) in Proposition B.2.

For $b \in (0, \underline{b}_B]$, from the lenders' break-even condition, in the Safe Zone, we have $(r + \delta)q_B(b) = q_B'(b)\dot{b} = q_B'(b)\left(\overline{C} + [r + \delta(1 - q_B(b))]b - y\right)$. Differentiating V_B in (4) and using this expression to substitute for $q_B'(b)$, we have for $b \in (0, \underline{b}_B]$

$$V_B'(b) = -q_B(b)\left(\frac{\overline{C} - \rho V_B(b)}{\overline{C} - [r + \delta(1 - q_B(b))]b - y}\right).$$

Hence, for $b \in (0, \underline{b}_B]$, the HJB is satisfied if and only if $\rho V_B(b) \geq y - [r + \delta(1 - q_B(b))]b$, which is the condition in equation (22).

For $b \in (\underline{b}_B, \overline{b}_B]$, we have $q_B(b) = \underline{q}$ and

$$(\rho + \lambda)V_B(b) = \overline{C} + \lambda\overline{V} - \left(\overline{C} + \lambda\overline{V} - (\rho + \lambda)\underline{V}\right)\left(\frac{\overline{C} - y + (r + \lambda)\underline{q}b}{\overline{C} - y + (r + \lambda)\underline{q}\overline{b}_B}\right)^{\frac{\rho+\lambda}{r+\lambda}}.$$

Note that $V_B(b)$ is concave in b, hence we need to check the condition at $b \to \overline{b}_B$. We have for $b \in (\underline{b}_B, \overline{b}_B]$

$$V_B'(b) \geq -\frac{\overline{C} + \lambda\overline{V} - (\rho + \lambda)\underline{V}}{\overline{C} - y + (r + \lambda)\underline{q}\overline{b}_B} = -1,$$

where the final equality uses the definition of $\overline{b}_B$; hence, for this region the optimality condition always holds.

By construction, for $b = \overline{b}_B$, condition (v) of Proposition B.1 is satisfied.

Note that as $V_B(\underline{b}_B) = \overline{V}$, the derivative of V_B is continuous at $\underline{b}_B$. The only point of non-differentiability is $b = 0$. In particular, note that $\lim_{b\downarrow 0} V_B'(b) = -\lim_{b\downarrow 0} q_B(b)(\overline{C} - \rho V_B(0))(\overline{C} - y)$. Hence, if $\lim_{b\downarrow 0} q_B(b) < 1$, then there is a convex kink at $b = 0$. This is consistent with condition (iii) in Proposition B.2.

Hence, the conditions of Proposition B.2 hold if and only if (22) holds. □

C.7 Proof of Proposition 4

Proof. There are three claims in the proposition:

Part (i). If a borrowing allocation is efficient, it must dominate the stationary allocation, hence

$$rP_B^\star(v) \geq y - \rho V$$

for any $V \geq \overline{V}$. From the expressions for $P_B^\star$ and V_B, this implies that:

$$\begin{aligned} V_B(b) &\geq \frac{y - rq_B(b)b}{\rho} \\ &= \frac{y - \frac{rq_B(b)}{r+\delta(1-q_B(b))}(r+\delta(1-q_B(b)))b}{\rho} \\ &\geq \frac{y - (r+\delta(1-q_B(b)))b}{\rho} \text{ for all } b \in [0, \underline{b}_B], \end{aligned}$$

where the last inequality follows from $rq_B(b) \leq r + \delta(1 - q_B(b))$ for all $\delta \geq 0$, $q_B(b) \leq 1$ and $b \geq 0$. And thus condition (22) is satisfied.

Part (ii). For $b \in [0, \underline{b}_B]$, Condition (22) becomes

$$\rho V_B(b) - (y - (r+\delta)b + \delta q_B(b)b) \geq 0.$$

Now, from the price equation (3), we have

$$\left(\frac{1 - q_B(b)}{1 - \underline{q}}\right)^{\frac{r}{r+\delta}} = \frac{\overline{C} - y + rp}{\overline{C} - y + r\underline{p}}, \tag{15}$$

where $\underline{p} \equiv P_B^\star(\overline{V}) = \underline{q}\underline{b}_B$. From this expression, we can define $q_B(b) = F(\delta, p)$, holding the other parameters constant. Recall that condition (22) is restricted to $b \in [0, \underline{b}_B]$; hence, the domain of interest for p is $[0, \underline{p}]$, which is independent of δ. We shall use the fact that

$$\frac{\partial F(\delta, p)}{\partial \delta} = \frac{1 - F(\delta, p)}{r + \delta}\left(\underline{q} + \ln\left(\frac{1 - \underline{q}}{1 - F(\delta, p)}\right)\right), \tag{16}$$

keeping in mind that $\underline{q} = (r+\delta)/(r+\delta+\lambda)$ and hence varies with δ.

Let $V_B^\star$ denote the inverse of $P_B^\star$. Recall that $V_B(b) = V_B^\star(q_B(b)b)$. Condition (22) can be written:

$$G(\delta, p) \equiv \rho V_B^\star(p) - y + (r+\delta)p/F(\delta, p) - \delta p \geq 0.$$

Taking the derivative with respect to δ, we have that:

$$\begin{aligned}
\frac{\partial G(\delta,p)}{\partial\delta} &= \frac{p}{F(\delta,p)} - p - \frac{(r+\delta)p}{F(\delta,p)^2}\frac{\partial F(\delta,p)}{\partial\delta} \\
&= \frac{p}{F(\delta,p)}\left(1 - F(\delta,p) - \frac{(r+\delta)}{F(\delta,p)}\frac{\partial F(\delta,p)}{\partial\delta}\right) \\
&= \frac{p(1-F(\delta,p))}{F(\delta,p)^2}\left(F(\delta,p) - \frac{(r+\delta)}{(1-F(\delta,p))}\frac{\partial F(\delta,p)}{\partial\delta}\right) \\
&= \frac{p(1-F(\delta,p))}{F(\delta,p)^2}\left(F(\delta,p) - \underline{q} - \ln\left(\frac{1-\underline{q}}{1-F(\delta,p)}\right)\right).
\end{aligned}$$

Note that $\partial G/\partial\delta \leq 0$ if

$$F(\delta,p) - \underline{q} - \ln\left(\frac{1-\underline{q}}{1-F(\delta,p)}\right) \leq 0.$$

For $p = \underline{p}$, $F(\delta,\underline{p}) = \underline{q}$, and this term is zero. Moreover, this expression is increasing in p as $\partial F/\partial p < 0$. Hence, $\partial G(\delta,p)/\partial\delta \leq 0$ for $p \in [0,\underline{p}]$. Thus, if $G(\delta_0,p) \geq 0$, then $G(\delta,p) \geq 0$ for $\delta \in [0,\delta_0]$.

Part (iii). The fact that saving is efficient implies

$$\frac{y-\rho\overline{V}}{r} > P_B^\star(\overline{V}) = \underline{q}\underline{b}_B,$$

where the last equality follows from the definition of $\underline{b}_B$. By continuity, there exists a $V_0 > \overline{V}$ such that

$$\frac{y-\rho V_0}{r} > P_B^\star(V_0) \equiv p_0 < \underline{p},$$

where the last inequality follows from the fact that $P_B^\star$ is strictly decreasing. As $V_0 = V_B^\star(p_0)$ by definition of $V_B^\star$ as the inverse of $P_B^\star$, this is equivalent to

$$\rho V_B^\star(p_0) < y - rp_0.$$

Evaluated at $p = p_0$, condition (22) is

$$G(\delta,p_0) = \rho V_B^\star(p_0) - y + \frac{rp_0}{F(\delta,p_0)} + p_0\delta\left(\frac{1}{F(\delta,p_0)} - 1\right) \geq 0. \tag{17}$$

Note that $\lim_{\delta\to\infty}\underline{q} = 1$, and hence $F(\delta,p) \geq \underline{q}$ also converges to 1. Hence, $\rho V_B^\star(p_0) - y + rp_0/F(\delta,p_0) \to \rho V_B^\star(p_0) - y + rp_0 < 0$. We now show that the last term in (17) converges to zero;

that is, $\delta(1-F(\delta,p_0)) \longrightarrow 0$. From the definition of F in (15), we have

$$\delta(1-F(\delta,p_0)) = \frac{\lambda\delta}{r+\delta+\lambda}\left(\underbrace{\frac{\overline{C}-y+rp_0}{\overline{C}-y+r\underline{p}}}_{<1}\right)^{\frac{r+\delta}{r}}.$$

As the ratio raised to the power $(r+\delta)/r$ is strictly less than one as $p_0 < \underline{p}$, the right-hand side goes to zero as $\delta \longrightarrow \infty$. Hence, there exists a δ_1 such that for all $\delta > \delta_1$, $\overline{G}(\delta,p_0) < 0$, violating the condition for the borrowing equilibrium.

□

C.8 Proof of Proposition 5

Proof. We proceed to show the necessity and sufficiency parts of the proposition independently.

The "only if" part. Toward a contradiction, suppose that $V_S(\underline{b}_S) < V_B(\underline{b}_S)$ (or equivalently $\underline{b}_B > \underline{b}_S$), and the conjectured saving equilibrium is indeed an equilibrium. First, note that $q_S(b) \in [\underline{q}, 1]$, as the government defaults only with the arrival of $V^D = \overline{V}$.

By construction, $\hat{V}(\underline{b}_S) = V_S(\underline{b}_S) = \overline{V}$. As V_S is strictly decreasing, we have $V_S(\underline{b}_B) < \overline{V} = V_B(\underline{b}_B)$. Hence, V_S and V_B do not intersect in $[\underline{b}_S, \underline{b}_B]$, and $b^I > \underline{b}_S$, and $V_S(\underline{b}_B) = \hat{V}(\underline{b}_B)$.

We also have for $b \geq \underline{b}_B$: $\hat{V}'(b) = -q_S(b) \leq -\underline{q} \leq V_B'(b)$, where the latter inequality uses a property of the borrowing allocation value function (shown in the proof of Proposition 3). This implies that $\hat{V}(b) < V_B(b)$ for all $b \geq \underline{b}_B$, and there is no point of intersection to generate b^I and $V_S = \hat{V}$ for all $b \in [\underline{b}_S, \overline{b}_S]$. Now at $\overline{b}_S$, we must have (as an equilibrium requirement) that $\hat{V}(\overline{b}_S) = \underline{V} < V_B(\overline{b}_S)$, where the latter inequality follows from the fact that $\hat{V} < V_B$ on this domain. Thus, $\overline{b}_S < \overline{b}_B$. However, we have

$$\begin{aligned}(\rho+\lambda)\underline{V} &= y - [r+\delta(1-\underline{q})]\overline{b}_B + \lambda\overline{V} \\ &< y - [r+\delta(1-\underline{q})]\overline{b}_S + \lambda\overline{V} \\ &\leq y - [r+\delta(1-q_S(\overline{b}_S))]\overline{b}_S + \lambda\overline{V} \\ &= (\rho+\lambda)\hat{V}(\overline{b}_S) = (\rho+\lambda)\underline{V},\end{aligned}$$

where the first line uses the definition of $\overline{b}_B$; the second line uses $\overline{b}_B > \overline{b}_S$; the third uses $q_S(b) \geq \underline{q}$; and the final two equalities use the fact that $\hat{V}(b)$ is the stationary value at price q_S and the definition of $\overline{b}_S$, respectively. Hence, we have generated a contradiction.

The "if" part. We first verify that V_S satisfies the conditions of Proposition B.2 and establish that q_S is a valid equilibrium price schedule.

First, consider the government's problem.

Condition (iv) and (v) of Proposition B.2 are satisfied by construction. For $b = \underline{b}_S$, condition (ii) of Proposition B.2 applies and is satisfied by construction.

For $b < \underline{b}_S$, the conjectured value function is differentiable. For the HJB to hold with $c = \overline{C}$ given that $q_S(b) = 1$ in this region, we require $V_S'(b) \geq -q_S(b) = -1$. On this domain, $V_S(b) = V_S^\star(b)$, where the latter is the inverse of the efficient solution $P_S^\star$. As $P_S^{\star\prime}(v) \leq -1$, we have $V_S'(b) \geq -1 = -q_S(b)$. Hence, $c = \overline{C}$ is indeed optimal, and the HJB holds with equality.

For $b \in (\underline{b}_S, b^I)$, $V_S(b) = \hat{V}(b)$, and thus V_S is differentiable and satisfies the HJB with equality by construction. Note that if $q_S(b) \geq \overline{q}$ (something we check below), then $\dot{b} \leq 0$ in (b_S, b^I) by equation (16) (consistent with the equilibrium conjecture that the government is saving in this region). This implies that $C_S(b) = \hat{C}(b) \leq \overline{C}$, and thus the conjectured policy function is a valid one (recall that we are assuming that $\underline{C}$ is sufficiently low and can thus be ignored as a constraint).

For $b \in (b^I, \overline{b}_S)$, $V_S(b) = V_B(b)$ and differentiability of V_B implies that V_S is differentiable. The proof of Proposition 3 establishes that the HJB holds with equality in this domain, given that $q^S(b) = \underline{q}$.

This confirms that condition (i) of Proposition B.2 holds.

If $b^I \in (\underline{b}_S, \overline{b}_S)$, $V_S(b^I) = V_B(b^I)$, and there is a potential point of non-differentiability at b^I. If $q_S(b^I) \geq \underline{q}$ (something that we check below), we have that this kink is convex. Thus, condition (iii) of Proposition B.2 holds.

Hence, given the conjectured q_S, the value function is a viscosity solution to the government's HJB equation.

Next, let us consider the price function. The only thing left to check is that $q^S(b) \in [\underline{q}, 1]$ for $b \in (\underline{b}_S, b^I)$, where $b^I \in (\underline{b}_B, \overline{b}_B)$. In this region, $q^S(b) \leq 1$ by equation (7). In addition,

$$\begin{aligned}(\rho + \lambda)V_S(b) &= y - \left[r + \delta(1 - q_S(b))\right]b + \lambda\overline{V} \\ &\geq (\rho + \lambda)V_B(b) \\ &\geq y - \left[r + \delta(1 - \underline{q})\right]b + \lambda\overline{V},\end{aligned}$$

where the first equality and second inequality follow from the equilibrium construction on $b \in (\underline{b}_S, b^I)$. The last inequality follows from the construction of $V_B(b)$ for $b \in (\underline{b}_B, \overline{b}_B)$. Comparison of the first and last lines establishes that $q_S(b) \geq \underline{q}$. □

C.9 Proof of Proposition 6

Proof. The fact that the efficiency of saving is a necessary condition for a saving equilibrium is established in the text. Turning to equation (25), multiply both sides of equation (24) by $\underline{q}$ to obtain the following necessary and sufficient condition:

$$\underline{q}P_S^\star(\overline{V}) \geq \underline{q}\underline{b}_B = P_B^\star(\overline{V}).$$

Using $\underline{q} = (r+\delta)/(r+\delta+\lambda)$ and the fact that $P_S^\star(\overline{V}) > P_B^\star(\overline{V})$, we solve for δ to obtain

$$\delta \geq \frac{\lambda P_B^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} - r = \frac{(r+\lambda)P_B^\star(\overline{V}) - rP_S^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} \geq 0,$$

where the last inequality is strict when $\rho > r$, as seen in the definition of $P_B^\star$. Thus, this is a necessary and sufficient condition for the saving equilibrium, proving the proposition. □

C.10 Proof of Proposition 7

Proof. Note that $P_B^\star(v)$ is increasing in $\overline{C}$. Hence,

$$\begin{aligned} P_B^\star(\overline{V}) &\leq \lim_{\overline{C}\to\infty} P_B^\star(\overline{V}) \\ &= \frac{y - \rho\overline{V} + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda} \\ &= \frac{rP_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda}. \end{aligned}$$

Then a sufficient condition for saving to be strictly efficient is

$$P_S^\star(\overline{V}) > \frac{rP_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V})}{r+\lambda},$$

or

$$\frac{\lambda}{\rho - r} > \frac{r(\overline{V} - \underline{V})}{y - \rho\overline{V}},$$

which is the last inequality in the proposition.

Similarly,

$$P_S^\star(\overline{V}) - P_B^\star(\overline{V}) \geq \frac{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})}{r+\lambda},$$

and

$$\begin{aligned} \frac{\lambda P_B^\star(\overline{V})}{P_S^\star(\overline{V}) - P_B^\star(\overline{V})} - r &\leq \frac{\lambda\left(rP_S^\star(\overline{V}) + (\rho - r)(\overline{V} - \underline{V})\right)}{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})} - r \\ &= \frac{(r+\lambda)(\rho - r)(\overline{V} - \underline{V})}{\lambda P_S^\star(\overline{V}) - (\rho - r)(\overline{V} - \underline{V})} \equiv \underline{\delta}. \end{aligned}$$

From Proposition 6, a sufficient condition for a saving equilibrium, given that saving is efficient, is that δ is greater than $\underline{\delta}$. Note that $\underline{\delta}$ is strictly positive and independent of $\overline{C}$.

For the borrowing equilibrium, we need to show that the condition in equation (22) is satisfied as $\overline{C}$ becomes arbitrarily large. Specifically, fix any $\delta = \overline{\delta} > \underline{\delta}$. Define

$$A(b) \equiv \rho V_B(b) - (y - [r + \delta(1 - q_B(b))]b),$$

where A implicitly depends on $\overline{C}$ and δ. Note by condition (22) in Proposition 3 that if $A(b) > 0$ on $[0, \overline{b}_B] \supseteq [0, \underline{b}_B]$, then a borrowing equilibrium exists.

To establish the properties of $A(b)$ as $\overline{C} \to \infty$, first note that $\overline{b}_B$ is independent of $\overline{C}$. In addition,

$$\lim_{\overline{C}\to\infty} V_B(b) = \underline{V} + \underline{q}(\overline{b}_B - b),$$

where we use the fact that $q_B(b) \to \underline{q}$ for $b \in [0, \overline{b}_B]$ as $\overline{C} \to \infty$. As the point-wise limit is continuous in b, and by Lemma B.2 $V_B(b)$ is monotonic given $\overline{C}$, the convergence is uniform on the compact set $[0, \overline{b}_B]$ (see Theorem A of Buchanan and Hildebrandt (1908)).

Similarly, for $b \in [0, \overline{b}_B]$,

$$\lim_{\overline{C}\to\infty} \{y - [r + \delta(1 - q_B(b))]b\} = y - [r + \delta(1 - \underline{q})]b = y - (r + \lambda)\underline{q}b.$$

Again, the convergence is uniform by the same logic.

Hence, $A(b)$ converges uniformly on $[0, \overline{b}_B]$ to

$$\overline{A}(b) \equiv \lim_{\overline{C}\to\infty} A(b) = \rho\underline{V} + \rho\underline{q}(\overline{b}_B - b) - (y - (r + \lambda)\underline{q}b).$$

We now establish that $\overline{A}(b) > 0$ for $b \in [0, \overline{b}_B]$. The linearity of $\overline{A}(b)$ in b implies that if the inequality holds for $b = 0$ and $b = \overline{b}_B$, it is satisfied for all intermediate points. For $b = 0$, we have

$$\begin{aligned} \overline{A}(0) &= \rho\underline{V} + \rho\underline{q}\overline{b}_B - y \\ &= \frac{(\rho - r - \lambda)(y - \rho\underline{V}) + \rho\lambda(\overline{V} - \underline{V})}{r + \lambda} \\ &= \frac{(\rho - r - \lambda)(y - \rho\overline{V}) + (\rho - r)\rho(\overline{V} - \underline{V})}{r + \lambda} > 0, \end{aligned}$$

where the second line uses the definition of $\underline{q}\overline{b}_B$ and the final inequality uses the condition in the proposition. Similarly,

$$\begin{aligned} \overline{A}(\overline{b}_B) &= \rho\underline{V} - (y - (r + \lambda)\underline{q}\overline{b}) \\ &= \lambda(\overline{V} - \underline{V}) > 0. \end{aligned}$$

Hence, $\min_{b\in[0,\overline{b}_B]} \overline{A}(b) = \min\{\overline{A}(0), \overline{A}(\overline{b})\} > 0$.

As $A \to \overline{A}$ uniformly on $[0, \overline{b}_B]$, for every $\epsilon > 0$, there exists an M such that for all $\overline{C} > M$, we have $A(b) > \overline{A}(b) - \epsilon$ for $b \in [0, \overline{b}_B]$. Setting $\epsilon < \min_{b\in[0,\overline{b}_B]} \overline{A}(b)$, we have $A(b) > 0$ for all

$b \in [0, \overline{b}_B]$ and $\overline{C} > M$. By Proposition 3, this is sufficient to establish the existence of a borrowing equilibrium for $\delta = \overline{\delta}$ when $\overline{C} > M$. By part (ii) of Proposition 4, we have a borrowing equilibrium for all $\delta \in [0, \overline{\delta}]$.

Combining results, there exists a non-empty interval $\Delta \equiv [\underline{\delta}, \overline{\delta}]$ and M such that for all $\overline{C} > M$ and $\delta \in \Delta$, both saving and borrowing equilibria coexist. □

C.11 Proof of Proposition 8

Proof. We first sketch out the borrowing equilibrium under the assumed policy. Let $\{V_B^P, q_B^P\}$ denote the equilibrium policy and price functions. The conjectured policy is for the government to borrow to $\overline{b}_B$, which is the endogenous limit in the borrowing equilibrium absent the policy. From (27), it is optimal for the bondholders to sell their bonds at price $q^* > \underline{q}$ as soon as $b = \underline{b}_B^P$, where the latter is defined by $V^P(\underline{b}_B^P) = \overline{V}$. That is, bondholders sell their bonds to the third party as soon as debt enters the Crisis Zone. We have

$$\begin{aligned} V_B^P(\overline{b}_B) &= \frac{y - [r + \delta(1 - q^*)]\overline{b}_B + \lambda \overline{V}}{\rho + \lambda} \\ &= \underline{V} + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda}. \end{aligned}$$

The last term reflects that the government rolls over debt at q^* rather than $\underline{q}$ once it reaches the borrowing limit. The expression assumes that the government defaults upon the arrival of $\overline{V}$. To see that this is optimal, note that the alternative of never defaulting yields the value

$$\frac{y - [r + \delta(1 - q^*)]\overline{b}_B}{\rho} \leq \frac{y - r\overline{b}_B}{\rho} < \frac{y - r\underline{b}_S}{\rho} = \overline{V}.$$

Facing $q_B^P(b) = q^*$ in the Crisis Zone, the government's value can be obtained from the HJB, and it is straightforward to verify that the first-order condition for $c = \overline{C}$ holds on this domain. As $q^* > \underline{q}$, $\underline{b}_B^P > \underline{b}_B$, where the latter is the benchmark borrowing equilibrium's threshold for the Safe Zone. Note as well that $q^* > \underline{q}$ implies that the third party takes a loss in expectation in the Crisis Zone.

For $b \in [0, \underline{b}_P]$, bondholders purchase debt at price $q_B^P(b)$ and collect r plus maturing principal until $b = \underline{b}_B^P$, at which point they sell at q^*. The equilibrium is recovered by solving the system of differential equations:

$$\begin{aligned} \rho V_B^P(b) &= \overline{C} + V_B^{P\prime}(b)\dot{b} \\ (r + \delta)q_B^P(b) &= r + \delta + q_B^{P\prime}(b)\dot{b} \\ \dot{b} &= \frac{\overline{C} + (r + \delta)b - y}{q_B^P(b)} - \delta b, \end{aligned}$$

with the boundary conditions $V_B^P(\underline{b}_B^P) = \overline{V}$ and $q_B^P(\underline{b}_B^P) = q^*$. Note that these equations are identical to those in the benchmark borrowing equilibrium except that the boundary condition $\underline{b}_B^P > \underline{b}_B$

and $q^* > \underline{q}$.

As is the case in the benchmark equilibrium, a necessary and sufficient condition for V_B^P to be a solution to the government's problem when facing q_B^P is

$$V_B^P(b) \geq \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho},$$

for all $b \in [0, \underline{b}_B]$. Following the same approach as in the proof of Proposition 7, we show that this inequality holds as $\overline{C} \to \infty$ uniformly over the full debt domain $[0, \overline{b}_B]$.

As $\overline{C} \to \infty$, we have for $b \in [0, \overline{b}_B]$,

$$\lim_{\overline{C}\to\infty} V_B^P(b) = V_B^P(\overline{b}_B) + q^*(\overline{b}_B - b)$$
$$\lim_{\overline{C}\to\infty} \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} = \frac{y - [r + \delta(1 - q^*)]b}{\rho}.$$

Recall from the proof of Proposition 7, that

$$\overline{A}(b) = \underline{V} + \underline{q}(\overline{b}_B - b) - \frac{y - [r + \delta(1 - \underline{q})]b}{\rho} \geq 0$$

under the conditions of the proposition. Note that this implies

$$\lim_{\overline{C}\to\infty} \left(V_B^P(b) - \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \right) = \overline{A}(b) + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} + (q^* - \underline{q})(\overline{b}_B - b) - \frac{\delta(q^* - \underline{q})}{\rho} b.$$

This expression is linear in b, and hence it is sufficient to verify the inequality at the endpoints $b = 0$ and $b = \overline{b}_B$. The fact that $\overline{A}(0) > 0$ and $q^* > \underline{q}$ implies that the limit is strictly positive at $b = 0$. For $b = \overline{b}_B$, we have

$$\begin{aligned}
&\overline{A}(\overline{b}_B) - \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\overline{b}_B \\
&= \frac{y - [r + \delta(1 - \underline{q})]\overline{b}_B + \lambda\overline{V}}{\rho + \lambda} - \frac{y - [r + \delta(1 - \underline{q})]\overline{b}_B}{\rho} + \frac{\delta(q^* - \underline{q})\overline{b}_B}{\rho + \lambda} - \frac{\delta(q^* - \underline{q})}{\rho}\overline{b}_B \\
&= \frac{-\lambda}{\rho(\rho + \lambda)} \left(y - [r + \delta(1 - q^*)]\overline{b}_B - \rho\overline{V} \right) = \frac{-\lambda}{\rho(\rho + \lambda)} \left(r(\underline{b}_S - \overline{b}_B) - \delta(1 - q^*)\overline{b}_B \right) > 0,
\end{aligned}$$

where the last inequality uses $\overline{b}_B > \underline{b}_S$. This completes the proof of part (i).

For part (ii), the saving equilibrium requires $V_B^P(\underline{b}_S) \le \overline{V}$. As $\overline{C} \to \infty$,

$$\begin{aligned} V_B^P(\underline{b}_S) &= V_B^P(\overline{b}_B) + q^*(\overline{b}_B - \underline{b}_S) \\ &= \frac{y - r\overline{b}_B + \lambda\overline{V}}{\rho + \lambda} + \overline{b}_B - \underline{b}_S - (1 - q^*)(\overline{b}_B - \underline{b}_S) - \frac{\delta(1 - q^*)\overline{b}_B}{\rho + \lambda} \\ &= \overline{V} + \frac{(\rho + \lambda - r)(\overline{b}_B - \underline{b}_S)}{\rho + \lambda} - (1 - q^*)\left(\overline{b}_B - \underline{b}_S + \frac{\delta\overline{b}_B}{\rho + \lambda}\right). \end{aligned}$$

As the second term is strictly positive, there exists a $\tilde{q} < 1$ such that this expression exceeds $\overline{V}$ for $q^* > \tilde{q}$, hence violating the necessary condition for a saving equilibrium.

□

C.12 Proof of Proposition 9

Proof. For part (i), note that in the saving equilibrium undistorted by policy, $q_S(b) = 1$ for $b \le \underline{b}_S$. Hence, imposing a price floor restricted to the Safe Zone does not alter the saving equilibrium, which exists by Proposition 7. Hence, the price floor is irrelevant under the saving equilibrium.

Using the notation introduced in the proof of Proposition 8, a necessary condition for the borrowing equilibrium under the policy is for $b \in [0, \underline{b}_B^P]$

$$V_B^P(b) \ge \frac{y - [r + \delta(1 - q_B^P(b))]b}{\rho} \ge \frac{y - [r + \delta(1 - q^*)]b}{\rho}.$$

Recall that in the construction of the borrowing equilibrium, $\underline{b}_B$ is defined by solving the HJB assuming $q_B(b) = \underline{q}$. Hence, $V_B^P(b) = V_B(b)$ for $b > \underline{b}_S$, as the policy is restricted to $b \in [0, \underline{b}_S]$. As $V_B(\underline{b}_S) < \overline{V}$ by the inequality of Proposition 7, we have $\underline{b}_B^P < \underline{b}_S$. For $b = \underline{b}_B^P$, we have

$$V_B^P(\underline{b}_B^P) = \overline{V} = \frac{y - r\underline{b}_S}{\rho} < \frac{y - r\underline{b}_B^P}{\rho},$$

where the first two equalities use the definitions of $\underline{b}_B^P$ and $\underline{b}_S$, respectively. Hence, there exists a $\hat{q} < 1$ such that

$$V_B^P(\underline{b}_B^P) < \frac{y - [r + \delta(1 - q^*)]\underline{b}_B^P}{\rho},$$

for $q^* > \hat{q}$, violating the necessary condition for a borrowing equilibrium. This proves part (ii).

□

Appendix D The Hybrid Equilibrium

In this appendix, we present a third type of competitive equilibrium, which we label the "hybrid" equilibrium because it combines features of both borrowing and saving equilibria. In particular, the government never saves, as in the borrowing equilibrium, but part of the Safe Zone is absorbing, as in the saving equilibrium. The main purpose of introducing the hybrid equilibrium is to show existence of a competitive equilibrium; in particular, we prove that if neither the borrowing nor the saving equilibrium exists, then a hybrid equilibrium exists. The equilibrium objects are depicted in Figure D.1 using the same parameters as in Figures 1 and 3.

More formally, given V_B in (4), define the threshold

$$V_B(b_H) = \frac{y - rb_H}{\rho}, \tag{18}$$

if such a threshold exists on the domain $[0, \underline{b}_B] \cap [0, \underline{b}_S]$. The equilibrium conjecture is that for $b \leq b_H$, the government borrows up to b_H and then remains there indefinitely. This behavior is similar to the Safe Zone policy in the saving equilibrium, but the threshold b_H may be strictly below $\underline{b}_S$. At b_H, given that $V_B(\underline{b}_H) = (y - rb_H)/\rho$, the government is indifferent to remaining at b_H at risk-free prices versus borrowing to the debt limit at the borrowing equilibrium price schedule. The conjecture is that for $b > b_H$, the government borrows. In a hybrid equilibrium, therefore, b_H is a stationary point that is stable from the left but not the right.

For $b < b_H$, we solve the government's HJB assuming $c = \overline{C}$ to obtain a candidate V_H on this domain, using the boundary condition $\rho V_H(\underline{b_H}) y - rb_H$. For $b > b_H$, the hybrid equilibrium coincides with the borrowing equilibrium. Setting $\overline{b}_H \equiv \overline{b}_B$, the hybrid equilibrium value function is therefore

$$V_H(b) = \begin{cases} \dfrac{\overline{C} - (\overline{C} + rb_H - y)\left(\frac{\overline{C} + rb - y}{\overline{C} + rb_H - y}\right)^{\frac{\rho}{r}}}{\rho} & \text{for } b \leq b_H \\ V_B(b) & \text{for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{19}$$

The associated price schedule is

$$q_H(b) = \begin{cases} 1 \text{ for } b \leq b_H \\ q_B(b) \text{ for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{20}$$

Finally, the policy function for consumption is

$$C_H(b) = \begin{cases} \overline{C} \text{ for } b < b_H \\ y - rb_H \text{ for } b = b_H \\ C_B \text{ for } b \in (b_H, \overline{b}_H]. \end{cases} \tag{21}$$

We state the following:

Proposition D.3. *Suppose neither the borrowing equilibrium nor the saving equilibrium exists. Specifically, suppose that $\underline{b}_S < \underline{b}_B$ and that there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b}$. Then a hybrid equilibrium exists.*

Proof. The conjectured price schedule q_H is consistent with the lenders' break-even condition given the assumed government policy. Thus, to establish the conditions of an equilibrium, it is sufficient to prove that V_H is a solution to the government's HJB.

(i) For $b \in [\underline{b}_S, \overline{b}_B]$: By premise, $\underline{b}_B > \underline{b}_S$. This implies that $\rho V_B(\underline{b}_S) > \overline{V} = y - r\underline{b}_S \geq y - [r + \delta(1 - q_B(\underline{b}_S))]\underline{b}_S$. For $b > \underline{b}_S$, we have $\rho\overline{V} > y - rb$. As $V_B \geq \overline{V}$ for $b \leq \underline{b}_B$, we have $V_B(b) \geq y - rb$ for $b \in [\underline{b}_S, \underline{b}_B]$. From the proof of Proposition 3, this implies that $V_H(b) = V_B(b)$ satisfies the government's HJB on this domain. The proof of Proposition 3 extends this to $b \in [\underline{b}_B, \overline{b}_B]$ as well.

(ii) For $b \leq \underline{b}_S$: Note that the premise implies there exists a $\hat{b} \in [0, \underline{b}_B]$ such that $\rho V_B(\hat{b}) < y - [r + \delta(1 - q_B(\hat{b}))]\hat{b} \leq y - r\hat{b}$. The above established that $\rho V_B(b) > y - rb$ for $b \in [\underline{b}_S, \overline{b}_B]$. Hence, $\hat{b} < \underline{b}_S$. By continuity, there exists a $b_H \in (\hat{b}, \underline{b}_S)$ such that $\rho V_B(b_H) = y - rb_H$. Note as well that this implies $V_B'(b_H) = \lim_{b \downarrow b_H} V_H'(b) \geq -r/\rho$. From the expression for V_H, we have $\lim_{b \uparrow b_H} V_H(b_H) = -1 \leq -r/\rho$. Hence, V_H is either differentiable or has a convex kink at b_H, satisfying the conditions for a solution to the government's HJB at b_H. For $b < b_H$, $V_H'(b) \geq -1$, implying that the HJB is satisfied on this domain as well. Finally, $V_H(b) > \overline{V}$ for $b \leq b_H$, rationalizing the government's non-default on this domain.

□

This establishes that at least one of the three types of equilibria always exists. We note that the hybrid may coexist with the other equilibria as well. In fact, as $\overline{C} \to \infty$, the condition for multiplicity presented in Proposition 7 also implies the existence of a hybrid equilibrium.

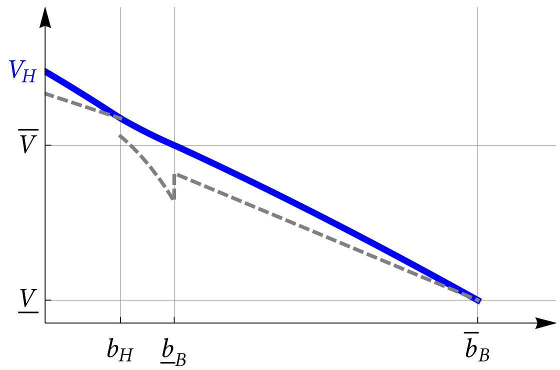
Appendix E Viscosity Solutions on Stratified Domains and the Proofs of Propositions B.1 and B.2

In this appendix, we establish the equivalence between the sequence problems and the viscosity solutions of the Hamilton-Jacobi-Bellman (HJB) equations. The two complications are that the objective and/or the dynamics are not necessarily continuous in the state variables. We rely on the results of Bressan and Hong (2007) (henceforth, BH) to establish the validity of the recursive formulation. This appendix introduces their environment and summarizes their core results. Relative to BH, we make minor changes in notation and consider a maximization problem while the original BH studies minimization. We then prove Propositions B.1 and B.2.

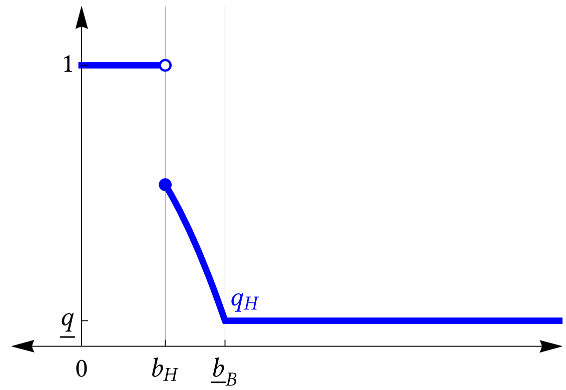
E.1 The Environment of Bressan and Hong (2007)

Let $X \subset \mathbb{R}^N$ denote the state space. In the benchmark BH environment, $X = \mathbb{R}^N$; however, they show how to restrict attention to an arbitrary subset by extending the dynamics and payoff functions to $\mathbb{R}^N$ such that the subset is an absorbing region. Let $\alpha(t) \in \mathcal{A}$ denote the control function, where $\mathcal{A}$ is the set of admissible controls. Dynamics of the state vector x are given by $\dot{x}(t) = f(x(t), \alpha(t))$.

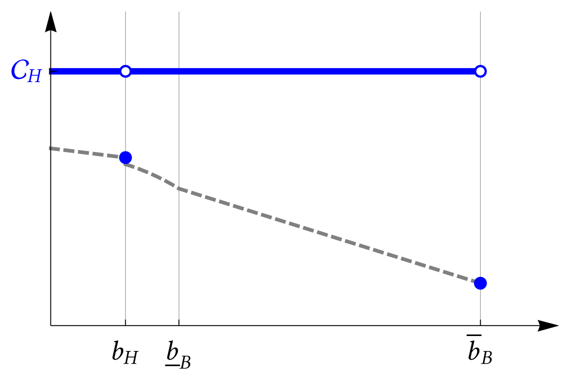
Figure D.1: Hybrid Equilibrium



(a) Policy functions



(b) q_H



(c) Price schedules

Given a discount factor β and a flow payoff $\ell(x,\alpha)$, the sequence problem is

$$W(\bar{x}) = \sup_{\alpha \in \mathcal{A}} \int_0^\infty \ell(x(t),\alpha(t))dt \tag{22}$$

$$\text{subject to} \begin{cases} x(0) & = \bar{x} \in X \\ \dot{x}(t) & = f(x(t),\alpha(t)). \end{cases}$$

The complication BH address is that f may not be continuous in x. In particular, assume there is a decomposition $X = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_M$ with the following properties. If $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_i = \emptyset$. In addition, if $\mathcal{M}_j \cap \overline{\mathcal{M}_k} \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}_k}$, where $\overline{\mathcal{M}_k}$ is the closure of $\mathcal{M}_k$.

BH's assumption $\boldsymbol{H1}$ ensures that dynamics are well behaved within $\mathcal{M}_i$:

Assumption. $\boldsymbol{H1}$: *For each $i = 1, \ldots, M$, there exists a compact set of controls $A_i \subset \mathbb{R}^m$, a continuous map $f_i : \mathcal{M}_i \times A_i \to \mathbb{R}^N$, and a payoff function ℓ_i, with the following properties:*

(a) At each $x \in \mathcal{M}_i$, all vectors $f_i(x,a)$, $a \in A_i$ are tangent to $\mathcal{M}_i$;

(b) $|f_i(x,a) - f_i(z,a)| \leq K_i|x - z|$, for some $K_i \in [0,\infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$;

(c) *Each payoff function ℓ_i is non-positive and $|\ell_i(x,a) - \ell_i(z,a)| \leq L_i|x-z|$, for some $L_i \in [0,\infty)$, for all $x, z \in \mathcal{M}_i$, $a \in A_i$;*[7]

(d) *We have $f(x,a) = f_i(x,a)$ and $\ell(x,a) = \ell_i(x,a)$ whenever $x \in \mathcal{M}_i$, $i = 1, ..., M$.*

The key assumption is (b); namely, that dynamics are Lipschitz continuous when confined to tangent trajectories. This does not restrict how the dynamics change when crossing the boundaries of $\mathcal{M}_i$.

Let $\mathcal{T}_{\mathcal{M}_i}(x)$ denote the cone of trajectories tangent to $\mathcal{M}_i$ at $x \in \mathcal{M}_i$:

$$\mathcal{T}_{M_i}(x) \equiv \left\{ y \in \mathbb{R}^N \middle| \lim_{h \to 0} \frac{\inf_{z \in \mathcal{M}_i} |x + hy - z|}{h} = 0 \right\}.$$

Part (a) of $\boldsymbol{H1}$ is equivalent to $f_i(x,a) \in \mathcal{T}_{\mathcal{M}_i}$ for all $x \in \mathcal{M}_i$, $a \in A_i$.

For $x \in \mathcal{M}_i$, let $\hat{F}(x) \subset \mathbb{R}^{N+1}$ denote the set of achievable dynamics and payoffs for the set of controls A_i:

$$\hat{F}(x) \equiv \{(\dot{x}, u) | \dot{x} = f_i(x,a), u \leq \ell_i(x,a), a \in A_i\}, \tag{23}$$

where i is such that $x \in \mathcal{M}_i$. To handle discontinuous dynamics, BH use results from differential inclusions. In particular, let $G(x)$ denote an extended set of feasible trajectories and payoffs:

$$G(x) \equiv \cap_{\epsilon > 0} \overline{co} \left\{ (\dot{x}, u) \in \hat{F}(x') \middle| |x - x'| < \epsilon \right\}, \tag{24}$$

where $\overline{co} S$ denotes the closed convex hull of a set S.

The next key assumption is that $G(x)$ does not contain additional trajectory-payoff pairs when restricted to tangent trajectories:

Assumption. $\boldsymbol{H2}$: *For every $x \in \mathbb{R}^N$, we have*

$$\hat{F}(x) = \left\{ (\dot{x}, u) \in G(x) | \dot{x} \in \mathcal{T}_{\mathcal{M}_i} \right\}. \tag{25}$$

BH define the Hamiltonian using $G(x)$ as the relevant choice set:

$$H(x,p) \equiv \sup_{(\dot{x},u) \in G(x)} \{u + p\dot{x}\}. \tag{26}$$

The corresponding HJB is

$$\beta w(x) = H(x, Dw(x)), \tag{27}$$

where D is the differential operator. BH define the following concepts:

Definition 1. *A continuous function w is a* ***lower solution*** *of* (27) *if the following holds: If $w - \varphi$ has a local maximum at x for some continuously differential φ, then*

$$\beta w(x) - H(x, D\varphi(x)) \leq 0. \tag{28}$$

[7] We strengthen part (c) to incorporate the Lipschitz continuity condition stated in BH equation (46).

Definition 2. *A continuous function w is an* ***upper solution*** *of* (27) *if the following holds: If $x \in \mathcal{M}_i$, and the restriction of $w - \varphi$ to $\mathcal{M}_i$ has a local minimum at x for some continuously differential φ, then*

$$\beta w(x) - \sup_{(\dot{x},u)\in G(x), \dot{x}\in T_{\mathcal{M}_i}} \{u + D\varphi \dot{x}\} \geq 0. \tag{29}$$

Definition 3. *A continuous function w, which is both an upper and a lower solution of* (27)*, is a* ***viscosity solution****.*

Note that the second definition differs from the first by restricting attention to $\mathcal{M}_i$ when describing the properties of $w-\varphi$, which relaxes the set of φ that satisfies the condition. However, the trajectories in the Hamiltonian are restricted to lie in the tangent set.[8] The added properties are the core distinction between the definition of viscosity solution used here versus the standard definition.[9]

With these definitions in hand, we summarize the core results of BH:

(i) (BH Theorem 1) If $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold, and there exists at least one trajectory with finite value, then the maximization problem admits an optimal solution.

(ii) (BH Proposition 1) Let assumptions $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold. If the value function W is continuous, then it is a viscosity solution of (27).

(iii) (BH Corollary 1) Let assumptions $\boldsymbol{H1}$ and $\boldsymbol{H2}$ hold. If the value function W is bounded and Lipschitz continuous, then W is the unique non-positive viscosity solution to (27).[10]

E.2 The Planner's Problem

To map problem (3) into BH, we make a few modifications and consider a generalized problem. First, we let the planner randomize when the government is indifferent to default or not. This helps to convexify the choice set. In particular, let $\pi(t) \in [0,1]$ be an additional choice, where $\pi(t)$ is the probability the government defaults when $\overline{V}$ arises and the current value is $\overline{V}$. It will always be efficient to set $\pi(t) = 0$ when $v(t) = \overline{V}$, and so this does not alter the solution to the planner's problem. We denote the set of possible paths, $\boldsymbol{\pi} = \{\pi(t) \in [0,1]\}_{t\geq 0}$, by $\boldsymbol{\Pi}$. The controls are $\boldsymbol{\alpha} = (\boldsymbol{c}, \boldsymbol{\pi}) \in \mathcal{A} \equiv \boldsymbol{C} \times \boldsymbol{\Pi}$.

Recall that in (3) the objective is discounted by the probability of repayment, $e^{-\lambda \int_0^t \mathbb{1}_{[v(s)<\overline{V}]} ds}$. Let us define $\Gamma(t)$ as follows:

$$\Gamma(t) \equiv \Gamma(0) e^{-\lambda \int_0^t (\pi(s)\mathbb{1}_{[v(s)=\overline{V}]} + \mathbb{1}_{[v(s)<\overline{V}]} ds}$$

[8] The fact that trajectories are restricted to T_{M_i} in the definition of an upper solution was unintentionally omitted in Bressan and Hong (2007) but is corrected in Bressan (2013).

[9] Note that we place the restriction on the upper solution while the original BH place it on the lower solution as we consider a maximization problem.

[10] BH state a weaker continuity condition than Lipschitz continuity (BH $\boldsymbol{H3}$) that is not necessary given our environment.

for some $\Gamma(0) \in (0,1]$. Note that $\Gamma(t)/\Gamma(0)$ is the discount factor in the original problem with $\pi = 0$. By adding $\Gamma(t)$ as an additional state variable, we will be able to keep track of the probability of survival in our recursive formulation.

Let $X = \mathbb{V} \times (0,1]$ denote the state space for $x = (v, \Gamma)$. Let $f(x,\alpha) = (\dot{v}, \dot{\Gamma})$ given the control $\alpha = (c, \pi)$:

$$f(x,\alpha) = \begin{cases} \dot{v} & = -c + \rho v - \mathbb{1}_{[v<\overline{V}]}\lambda \left[\overline{V} - v\right] \\ \dot{\Gamma} & = -\lambda \left[\pi \mathbb{1}_{[v=\overline{V}]} + \mathbb{1}_{[v<\overline{V}]}\right]\Gamma. \end{cases} \tag{30}$$

The flow value must be non-positive. We therefore subtract the constant $(y - \underline{C})/r$ from the value. To convert this into a flow payoff, let

$$\ell(x,a) \equiv \Gamma(y-c) - (y - \underline{C}) \leq 0,$$

where the inequality uses $y > \underline{C}$ and $\Gamma \leq 1$. Note that ℓ is Lipschitz continuous in x.

Hence, we consider the following problem, where $x(t) \equiv (v(t), \Gamma(t))$:

$$\tilde{P}(v,\Gamma) = \sup_{\alpha \in \mathcal{A}} \int_0^\infty e^{-rt} \ell(x(t), \alpha(t)) \mathrm{d}t \tag{31}$$
$$\text{subject to } \begin{cases} (v(0), \Gamma(0)) & = (v, \Gamma) \\ (\dot{v}(t), \dot{\Gamma}(t)) & = f(x(t), \alpha(t)). \end{cases}$$

We then have a one-to-one mapping between $\tilde{P}$ and $P^\star$:

$$\tilde{P}(v,\Gamma) = \Gamma P^\star(v) - (y - \underline{C})/r. \tag{32}$$

As $P^\star$ is bounded and $\Gamma \in (0,1]$, $\tilde{P}$ is bounded. Similarly, $\tilde{P}$ is Lipschitz continuous in the state vector (v, Γ).

We now verify the conditions of BH. Define five regions of the state space:

$$\begin{aligned} \mathcal{M}_1 &\equiv \{\underline{V}\} \times (0,1] \\ \mathcal{M}_2 &\equiv (\underline{V}, \overline{V}) \times (0,1] \\ \mathcal{M}_3 &\equiv \{\overline{V}\} \times (0,1] \\ \mathcal{M}_4 &\equiv (\overline{V}, V_{max}) \times (0,1] \\ \mathcal{M}_5 &\equiv \{V_{max}\} \times (0,1]. \end{aligned}$$

Let A_i denote the controls that produce trajectories that are tangent to $\mathcal{M}_i$:[11]

$$A_i \equiv \{(c,\pi)|c \in [\underline{C},\overline{C}], \pi \in [0,1], \dot{x} \in \mathcal{T}_{M_i}\}$$
$$= \begin{cases} \{\rho\underline{V} - \lambda(\overline{V}-\underline{V})\} \times [0,1] \text{ if } i=1 \\ \{\rho\overline{V}\} \times [0,1] \text{ if } i=3 \\ \{\rho V_{max}\} \times [0,1] \text{ if } i=5 \\ [\underline{C},\overline{C}] \times [0,1] \text{ otherwise.} \end{cases} \tag{33}$$

Within each $\mathcal{M}_i$, the dynamics f are Lipschitz continuous in x for all $a \in A_i$. It is straightforward to verify that we satisfy Assumption $\boldsymbol{H1}$.

Let us now verify Assumption $\boldsymbol{H2}$. There two cases:

Case 1: $i \in \{2,4\}$. In this case, $G(x) = \hat{F}(x)$, and BH Assumption $\boldsymbol{H2}$ is straightforward to verify.

Case 2: $i \in \{1,3,5\}$. We show the $i=3$ case (as the others are similar). We have[12]

$$\hat{F}(x) = \Big\{(\dot{x},u)|\dot{v}=0, \dot{\Gamma} = -\pi\lambda\Gamma, u \le \ell(x,(\rho\overline{V},\pi)), \pi \in [0,1]\Big\} \tag{34}$$
$$= \Big\{(\dot{x},u)|\dot{v}=0, \dot{\Gamma} \in [-\lambda\Gamma,0], u \le \Gamma(y-\rho\overline{V}) - (y-\underline{C})\Big\}$$
$$= \{0\} \times [-\lambda\Gamma,0] \times (-\infty, \Gamma(y-\rho\overline{V}) - (y-\underline{C})]. \tag{35}$$

Let $x' = (v',\Gamma')$ be in the neighborhood of $x = (\overline{V},\Gamma)$. We have

$$\hat{F}(x') = \Big\{(\dot{x},u)\Big| \quad \dot{v} = -c + \rho v' - \lambda \mathbb{1}_{\{v'<\overline{V}\}}(\overline{V}-v'), \quad \dot{\Gamma} \in [-\lambda\mathbb{1}_{\{v'<\overline{V}\}}\Gamma, 0], \quad u \le \Gamma(y-c) - (y-\underline{C}), c \in [\underline{C},\overline{C}]\Big\}.$$

We have that

$$\cup_{|x'-x|\le\epsilon}\hat{F}(x') \subseteq R(x,\epsilon) \equiv \Big\{\dot{v} = -c+\theta, \quad \dot{\Gamma} = [-\lambda(\Gamma+\epsilon),0], \quad u \le (\Gamma+\epsilon-1)y - (\Gamma-\epsilon)c + \underline{c}, \quad \theta \in [\rho(\overline{V}-\epsilon) - \lambda\epsilon, \rho(\overline{V}+\epsilon)] \quad c \in [\underline{C},\overline{C}]\Big\}.$$

[11]For $i = 1,3,5$, the tangent trajectories set $\dot{v} = 0$. Otherwise, they are the full set of trajectories.
[12]Note this is the only case where the choice of π is relevant.

Note that $R(x,\epsilon)$ is convex and $G(x) = \cap_{\epsilon>0} R(x,\epsilon)$. Also note that

$$\hat{F}(x) = \{(\dot{x}, u) \in G(x) | \dot{x} \in \mathcal{T}_{\mathcal{M}_3}\},$$

where the equivalence uses the definitions of G, $\hat{F}$, and the tangent trajectories $\mathcal{T}_{\mathcal{M}_3}$. This verifies BH $\boldsymbol{H2}$ for $\mathcal{M}_3$.

Similar steps hold for $i = 1$ and 5, verifying Assumption $\boldsymbol{H2}$ for all domains.[13]

As noted above, $\tilde{P}$ is bounded and Lipschitz continuous. Hence, by BH Corollary 1, it is the unique viscosity solution with such properties for the HJB:

$$r\tilde{P}(v,\Gamma) = H((v,\Gamma),(\tilde{P}_V,\tilde{P}_\Gamma)) \equiv \sup_{(c,\pi)\in[\underline{C},\overline{C}]\times[0,1]} \left\{\Gamma(y-c) - (y-\underline{C}) + \tilde{P}_v\dot{v} + \tilde{P}_\Gamma\dot{\Gamma}\right\}, \tag{36}$$

where $\dot{v}$ and $\dot{\Gamma}$ obey equation (30). Here, we have used the fact that $G(x)$ contains the full set of trajectories generated by $c \in [\underline{C},\overline{C}]$ and $\pi \in [0,1]$. Note that it is optimal to set π to 0, and thus we can ignore this choice in the Hamiltonian in what follows. We shall use the fact that H is convex in $\tilde{P}_v$.

E.3 Proof of Proposition B.1

Proof. Suppose that $p(v)$ satisfies the conditions in the proposition. We shall show that $\tilde{p}(v,\Gamma) \equiv \Gamma p(v) - (y-\underline{C})/r$ is a viscosity solution of (36). $\tilde{p}$ is differentiable in Γ at all points, and in v at points where $p(v)$ is differentiable. We now check the conditions for a viscosity solution. We proceed by checking on each domain $\mathcal{M}_i$.

(i) $(v,\Gamma) \in \mathcal{M}_2 \cup \mathcal{M}_4$

As p is differentiable on this part of the domain, by condition (i) of the proposition, we have

$$\begin{aligned} rp(v) &= \sup_{c\in[\underline{C},\overline{C}]} \left\{y - c + p'(v)\dot{v} + \mathbb{1}_{[v<\overline{V}]}p(v)\right\} \\ &= \sup_{c\in[\underline{C},\overline{C}]} \left\{y - c + \Gamma^{-1}\tilde{p}_v\dot{v} + \Gamma^{-1}\tilde{p}_\Gamma\dot{\Gamma}\right\}, \end{aligned}$$

where the second line uses $\tilde{p}_v = \Gamma p'(v)$ and $\tilde{p}_\Gamma\dot{\Gamma}/\Gamma = -\lambda\mathbb{1}_{[v<\overline{V}]}p$. Multiplying through by $\Gamma \in (0,1]$ and subtracting $(y-\underline{C})/r$ from both sides yields

$$\begin{aligned} r\tilde{p}(v) = r(\Gamma p(v) - (y-\underline{C})) &= \sup_{c\in[\underline{C},\overline{C}]} \left\{\Gamma(y-c) - (y-\underline{C})/r + \tilde{p}_v\dot{v} + \tilde{p}_\Gamma\dot{\Gamma}\right\} \\ &= H((v,\Gamma),(\tilde{p}_v,\tilde{p}_\Gamma)). \end{aligned}$$

Hence, $\tilde{p}$ satisfies (36).

Now consider a point of non-differentiability $\tilde{v}$. As $(v,\Gamma) \notin \mathcal{M}_3$, $\tilde{v} \neq \overline{V}$, and hence condition (iii) of the proposition is applicable. Condition (iii) states that $p^-_{\tilde{v}} \equiv \lim_{v\uparrow\tilde{v}} p'(v) <$

[13]For $v = \underline{V}$, we extend the dynamics to both sides of $\underline{V}$ by setting $\dot{v} = -c + \rho v - \lambda(\overline{V} - v)$ in the neighborhood $v < \underline{V}$ and ℓ arbitrarily low. Thus, the dynamics are continuous at $x = (\underline{V},\Gamma)$. Similarly for $v = V_{max}$, we set $\dot{v} = -c + \rho v$.

$\lim_{v\downarrow\tilde{v}} p'(v) \equiv p^+_{\tilde{v}}$). Hence, there is a convex kink. In this case, the lower solution does not impose additional conditions, leaving the conditions for an upper solution to be verified. Suppose φ is differentiable and $\tilde{p} - \varphi$ has a local minimum at $(\tilde{v}, \Gamma)$. Then $\varphi_v \in [p^-_{\tilde{v}}, p^+_{\tilde{v}}]$. As $\tilde{p}$ is differentiable in Γ, we have $\varphi_\Gamma = \tilde{p}_\Gamma$. Note that

$$r\tilde{p}(\tilde{v}) = H((\tilde{v}, \Gamma), (p^-_{\tilde{v}}, \tilde{p}_\Gamma)) = H((\tilde{v}, \Gamma), (p^+_{\tilde{v}}, \tilde{p}_\Gamma)), \tag{37}$$

as the HJB holds with equality at points of differentiability in the neighborhood of $\tilde{v}$, and using the continuity of H.

Note that there exists $\alpha \in [0, 1]$ such that $\varphi_v = \alpha p^+_{\tilde{v}} + (1 - \alpha)p^-_{\tilde{v}}$. The convexity of H with respect to φ_v implies that

$$\begin{aligned} H((\tilde{v}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= H((\tilde{v}, \Gamma), (\alpha p^+_{\tilde{v}} + (1 - \alpha)p^-_{\tilde{v}}, \varphi_\Gamma)) \\ &\leq \alpha H((\tilde{v}, \Gamma), (p^+_{\tilde{v}}, \varphi_\Gamma)) + (1 - \alpha)H((\tilde{v}, \Gamma), (p^-_{\tilde{v}}, \varphi_\Gamma)) \\ &= r\tilde{p}(\tilde{v}), \end{aligned}$$

where the last equality uses (37) and $\varphi_\Gamma = \tilde{p}_\Gamma$. Hence, $\tilde{p}(\tilde{v})$ satisfies the conditions of an upper solution.

(ii) $(v, \Gamma) \in \mathcal{M}_3 = \{\overline{V}\} \times (0, 1]$

Turning to $v = \overline{V}$, we redefine $p^+_v \equiv \lim_{v\downarrow\overline{V}} p'(v)$ and $p^-_v \equiv \lim_{v\uparrow\overline{V}} p'(v)$. Given the continuity of p and the fact that it satisfies the HJB in the neighborhood of $\overline{V}$ with equality, we have

$$\begin{aligned} rp(\overline{V}) &= \sup_{c\in[\underline{C},\overline{C}]} \{y - c + p^+_v \dot{v}\} \\ &= \sup_{c\in[\underline{C},\overline{C}]} \{y - c + p^-_v \dot{v} - \lambda p(\overline{V})\}, \end{aligned} \tag{38}$$

where $\dot{v} = -c + \rho\overline{V}$. As setting $c = \rho\overline{V}$ is always feasible, this implies $rp(\overline{V}) \geq (y - \rho\overline{V}) \geq 0$.

To verify that $\tilde{p}$ is a viscosity solution to (26), note that if $\tilde{p}$ is differentiable, then it satisfies the HJB with equality by condition (i) of the proposition.

If it is not differentiable, we consider convex and concave kinks in turn.

Suppose that $p^-_v < p^+_v$. Then the conditions for a lower solution do not impose any restrictions. For an upper solution, consider a φ such that $\tilde{p} - \varphi$ has a local minimum at $(\overline{V}, \Gamma)$. Again, $\varphi_\Gamma = \tilde{p}_\Gamma = p(\overline{V})$. Recall that for an upper solution, we need only consider trajectories that are in $\mathcal{T}_{\mathcal{M}_3}$, that is, $\dot{v} = 0$ and thus $c = \rho\overline{V}$. Hence:

$$\begin{aligned}
r\tilde{p}(\overline{V}) &= r\Gamma p(\overline{V}) - (y - \underline{C}) \\
&\geq \Gamma(y - \rho\overline{V}) - (y - \underline{C}) \\
&= \sup_{c=\rho\overline{V}} \left\{ \Gamma(y-c) - (y - \underline{C}) + \varphi_v \underbrace{\left(-c + \rho\overline{V}\right)}_{\dot{v}} \right\} \\
&= \sup_{c=\rho\overline{V}, \pi\in[0,1]} \left\{ \Gamma(y-c) - (y - \underline{C}) + \varphi_v \underbrace{\left(-c + \rho\overline{V}\right)}_{\dot{v}} - \underbrace{p(\overline{V}) \times \pi\lambda\Gamma}_{\varphi_\Gamma \times \dot{\Gamma}} \right\},
\end{aligned}$$

where the last equality uses $p(\overline{V}) \geq 0$. Note that final term is the Hamiltonian maximized along tangent trajectories in $\mathcal{T}_{\mathcal{M}_3}$. Thus, the conditions of an upper solution are satisfied.

For the lower solution, we must consider the case in which $\tilde{p} - \varphi$ has a local maximum at $(\overline{V}, \Gamma)$. This requires $p_v^- \geq p_v^+$ and $\varphi_v \in [p_v^+, p_v^-]$. Again, as $\tilde{p}$ is differentiable with respect to Γ, we have $\varphi_\Gamma = \tilde{p}_\Gamma = p(\overline{V})$.

If $p_v^+ \leq -1$, then condition (ii) in the proposition implies that

$$\begin{aligned}
r\tilde{p}(\overline{V}) &= \Gamma(y - \rho\overline{V}) - (y - \overline{C}) \\
&\leq \sup_{c\in[\underline{C},\overline{C}], \pi\in[0,1]} \left\{ \Gamma(y-c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho\overline{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi\lambda\Gamma)}_{\dot{\Gamma}} \right\} \\
&= H((\overline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)),
\end{aligned}$$

where the second to the last line follows from $\varphi_\Gamma = p(\overline{V}) \geq 0$. Hence, $\tilde{p}(\overline{V}) = \Gamma p(\overline{V}) - (y - \underline{C})/r$ satisfies the condition for a lower solution when $p_v^+ \leq -1$.

Alternatively, if $p_v^+ > -1$, then

$$\begin{aligned}
rp(\overline{V}) &= \sup_{c\in[\underline{C},\overline{C}]} \{y - c + p_v^+(-c + \rho\overline{V})\} \\
&= y - \underline{C} + p_v^+(\rho\overline{V} - \underline{C}) \\
&\leq y - \underline{C} + \varphi_v(\rho\overline{V} - \underline{C}),
\end{aligned}$$

for $\varphi_v \geq p_v^+$ as $\rho\overline{V} > \underline{C}$. Hence,

$$r\tilde{p}(\overline{V}) \leq \sup_{c\in[\underline{C},\overline{C}], \pi\in[0,1]} \left\{ \Gamma(y-c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho\overline{V})}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\pi\lambda\Gamma)}_{\dot{\Gamma}} \right\}$$

for $\varphi_v \in [p_v^+, p_v^-]$ and $\varphi_\Gamma = p(\overline{V})$, satisfying the condition for a lower solution.

(iii) $(v, \Gamma) \in \mathcal{M}_1 = \{\underline{V}\} \times (0, 1]$

For $v = \underline{V}$, the condition for $\tilde{p}$ to be an upper solution is

$$r\tilde{p}(\underline{V}, \Gamma) \geq \Gamma \left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) - (y - \underline{C}) - \lambda p(\underline{V})\Gamma,$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v} = 0$. As $\tilde{p}$ satisfies the HJB with equality in the neighborhood of $\underline{V}$, we have

$$\begin{aligned} p(\underline{V}, \Gamma) = \lim_{v \downarrow \underline{V}} r\tilde{p}(v, \Gamma) &= \lim_{v \downarrow \underline{V}} H((v, \Gamma), (\Gamma p'(v), p(v))) \\ &\geq \lim_{v \downarrow \underline{V}} \left\{ \Gamma \left(y - \rho v + \lambda(\overline{V} - v) \right) - (y - \underline{C}) - \lambda p(v)\Gamma, \right\} \\ &= \Gamma \left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) - (y - \underline{C}) - \lambda p(\underline{V})\Gamma. \end{aligned}$$

Hence, $\tilde{p}$ is an upper solution.

Turning to the lower solution, suppose $\tilde{p} - \varphi$ has a local maximum at $(\underline{V}, \Gamma)$. As $\underline{V}$ is at the boundary of the state space, this implies $\varphi_v \geq \tilde{p}_v(\underline{V}, \Gamma)$ and $\varphi_\Gamma = p(\underline{V})$. A lower solution requires

$$r\tilde{p}(\underline{V}, \Gamma) \leq H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)).$$

Suppose $p'(\underline{V}) < -1$. Then, condition (iv) of the proposition implies

$$\begin{aligned} r\tilde{p}(\underline{V}, \Gamma) &= r\Gamma p(\underline{V}) - (y - \underline{C}) \\ &= r\Gamma \left(\frac{y - \rho \underline{V} + \lambda(\overline{V} - \underline{V})}{r + \lambda} \right) - (y - \underline{C}) \\ &= \left(1 - \frac{\lambda}{r + \lambda} \right) \Gamma \left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) - (y - \underline{C}) \\ &\leq \sup_{c \in [\underline{C}, \overline{C}]} \left\{ \Gamma(y - c) - (y - \overline{C}) + \varphi_v \underbrace{(-c + \rho \underline{V} - \lambda(\overline{V} - \underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda\Gamma)}_{\dot{\Gamma}} \right\} \\ &= H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)), \end{aligned}$$

where the inequality uses

$$\begin{aligned} -\varphi_\Gamma \lambda \Gamma &= -p(\underline{V})\lambda\Gamma \\ &= -\left(y - \rho \underline{V} + \lambda(\overline{V} - \underline{V}) \right) \frac{\lambda}{r + \lambda} \Gamma. \end{aligned}$$

This verifies that $\tilde{p}$ is a lower solution if $p'(\underline{V}) < -1$.

Turning to $p'(\underline{V}) \geq -1$,

$$
\begin{aligned}
&H((\underline{V},\Gamma),(\varphi_v,\varphi_\Gamma)) = \\
&\quad = \sup_{c\in[\underline{C},\overline{C}]} \left\{ \Gamma(y-c) - (y-\overline{C}) + \varphi_v \underbrace{(-c+\rho\underline{V} - \lambda(\overline{V}-\underline{V}))}_{\dot{v}} + \varphi_\Gamma \underbrace{(-\lambda\Gamma)}_{\dot{\Gamma}} \right\} \\
&\quad = \Gamma(y-\underline{C}) - (y-\overline{C}) + \varphi_v \underbrace{(-\underline{C}+\rho\underline{V}-\lambda(\overline{V}-\underline{V}))}_{>0} + \varphi_\Gamma(-\lambda\Gamma) \\
&\quad \geq \Gamma(y-\underline{C}) - (y-\overline{C}) + \tilde{p}_v(\underline{V},\Gamma)(-\underline{C}+\rho\underline{V}-\lambda(\overline{V}-\underline{V})) + \varphi_\Gamma(-\lambda\Gamma) \\
&\quad = H((\underline{V},\Gamma),(\tilde{p}_v,\tilde{p}_\Gamma)) = r\tilde{p}(\underline{V},\Gamma),
\end{aligned}
$$

where the second equality uses the fact that $\underline{C}$ is optimal when $\varphi_v \geq \Gamma p'(v) \geq -\Gamma$; the inequality uses the fact that $\varphi_v \geq \tilde{p}_v$ and the term multiplying φ_v is positive; and the last line uses the continuity of the Hamiltonian and the value function, and that $\underline{C}$ is optimal given $p'(\underline{V}) \geq -1$. This verifies that $\tilde{p}$ is a lower solution if $p'(\underline{V}) \geq -1$.

(iv) $(v,\Gamma) \in \mathcal{M}_5 = \{V_{max}\} \times (0,1]$

For $v = V_{max}$, the condition for $\tilde{p}$ to be an upper solution is

$$r\tilde{p}(V_{max},\Gamma) \geq \Gamma\left(y-\rho V_{max}\right) - (y-\underline{C}),$$

where the right-hand side is the Hamiltonian evaluated at $\dot{v}=0$. As $\tilde{p}$ satisfies the HJB with equality in the neighborhood of V_{max}, we have

$$
\begin{aligned}
r\tilde{p}(V_{max},\Gamma) = \lim_{v\uparrow V_{max}} r\tilde{p}(v,\Gamma) &= \lim_{v\uparrow V_{max}} H((v,\Gamma),(\Gamma p'(v),p(v))) \\
&\geq \lim_{v\uparrow V_{max}} \left\{\Gamma\left(y-\rho v\right) - (y-\underline{C})\right\} \\
&= \Gamma\left(y-\rho V_{max}\right) - (y-\underline{C}).
\end{aligned}
$$

Hence, $\tilde{p}$ is an upper solution.

For the lower solution, suppose $\tilde{p}-\varphi$ has a local maximum at (V_{max},Γ). This implies $\varphi_v \leq \tilde{p}_v = \Gamma p'(V_{max})$ and $\varphi_\Gamma = p(V_{max})$. The condition for a lower solution is

$$
\begin{aligned}
r\tilde{p}(V_{max},\Gamma) &\leq \sup_{c\in[\underline{C},\overline{C}]} \left\{ \Gamma(y-c) - (y-\overline{C}) + \varphi_v \underbrace{(-c+\rho V_{max})}_{\dot{v}} \right\} \\
&= H((\underline{V},\Gamma),(\varphi_v,\varphi_\Gamma)).
\end{aligned}
$$

By condition (v) of the proposition, we have $p'(V_{max}) \leq -1$, implying that $\varphi_v \leq -\Gamma$. Hence,

$c = \overline{C}$ achieves the optimum in $H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma))$. That is,

$$\begin{aligned} H((\underline{V}, \Gamma), (\varphi_v, \varphi_\Gamma)) &= \Gamma(y - \overline{C}) - (y - \overline{C}) + \varphi_v \underbrace{(-\overline{C} + \rho V_{max})}_{\dot{v}} \\ &\geq \Gamma(y - \overline{C}) - (y - \overline{C}) + \tilde{p}_v(-\overline{C} + \rho V_{max}) \\ &= r\tilde{p}(V_{max}, \Gamma), \end{aligned}$$

where the inequality uses $\varphi_v \geq \tilde{p}_v$ and $\overline{C} \geq \rho V_{max}$; and the final line uses continuity of H and $\tilde{p}$. Hence, $\tilde{p}$ is a lower solution at (V_{max}, Γ).

We have shown that $\tilde{p}$ implied by a p satisfying the conditions of the proposition is a viscosity solution of the planner's problem. □

E.4 The Competitive Equilibrium

This section maps the government's problem into the BH framework.

Let us first define the following operator T that takes as an input a candidate value function, $\tilde{V}(b)$, assumed to be bounded and Lipschitz continuous, and a debt dynamics function $f(b, c)$ that embeds the price function, $q(b)$:

$$T\tilde{V}(b) = \int_0^\infty e^{-(r+\lambda)t} \left(c(t) + \lambda D(b(t)|\tilde{V}) \right) \tag{39}$$

subject to:

$$\dot{b}(t) = f(b(t), c(t))$$

$$b(0) = b,$$

where

$$D(b|\tilde{V}) \equiv \mathbb{1}_{[\tilde{V}(b) < \overline{V}]}(\overline{V} - \tilde{V}(b)).$$

The government's equilibrium value function is a fixed point of this operator. We shall map the right-hand side problem into the BH framework and use recursive techniques to solve the optimization. Toward this goal, let

$$\ell(b, c) \equiv c + \lambda D(b|\tilde{V}).$$

Note that $\ell(b, c)$ so defined is Lipschitz continuous and bounded. To be consistent with BH, we also need a non-positive ℓ. This can be achieved by subtracting the maximum value of ℓ. Rather than carrying this notation through, we proceed with the objects defined above, recognizing that all flow utilities and values can be appropriately translated (as we did explicitly in the planning problem).

Turning to the dynamics, $f(b, c)$, suppose the government faces a closed, convex domain $\boldsymbol{B}$ and an equilibrium price schedule $q : \boldsymbol{B} \to [\underline{q}, 1]$ that is differentiable almost everywhere with $|q'(b)| < M < \infty$.

Let $b_0 \equiv -\overline{a}$; $b_1, ..., b_N$ denote the N points of non-continuity in q; and $b_{N+1} \equiv \overline{b}$. We consider equilibria in which $\limsup_{b \to b_n} q(b) = q(b_n)$, as our tie-breaking rule is that the government chooses the action that maximizes the price when indifferent.

To define the domains, let $\mathcal{M}_n \equiv (b_{n-1}, b_n)$, $n = 1, ..., N+1$, be the open sets on which q is differentiable. Let $\mathcal{M}_{N+1+n} \equiv \{b_n\}$, $n = 1, ..., N$ be the isolated points of non-differentiability. Finally, we have the endpoints of the domain: $\{-\overline{a}\}$ and $\{\overline{b}\}$.

In the neighborhood of a discontinuity, we rule out repurchases at the "low price" (see footnotes 1 and 4). We do this while ensuring the continuity of dynamics. Specifically, let $\Delta > 0$ be arbitrarily small; and in particular, $\Delta < \inf_n |b_n - b_{n-1}|/2$. Define $\alpha(b) \equiv \min\{|b - b_n|/\Delta, 1\}$, where b_n is the closest point of non-differentiability to b. Note that $\alpha(b) \in [0, 1]$, and equals one if $|b - b_n| \geq \Delta$. Debt dynamics are given by

$$f(b,c) = \begin{cases} \frac{c-y+(r+\delta)b}{q(b)} - \delta b & \text{if } c \geq y - (r+\delta)b \\ \frac{c-y+(r+\delta)b}{\alpha(b)q(b)+(1-\alpha(b))q(b_n)} - \delta b & \text{if } c < y - (r+\delta)b. \end{cases} \tag{40}$$

Note that $f(b,c)$ is Lipschitz continuous in b and c *within the domains* $\mathcal{M}_n$.

On the open sets $\mathcal{M}_n$, $n = 1, ..., N+1$, any $c \in A_n \equiv [\underline{C}, \overline{C}]$ results in a tangent trajectory. For $n > N+1$, $c \in A_n \equiv y - [r + \delta(1 - q(b_n))]b_n$ is the singleton set that generates a tangent trajectory to the isolated point $\mathcal{M}_n$. Hence, BH assumption $\boldsymbol{H1}$ is satisfied.

Following BH, define

$$\hat{F}(b) \equiv \left\{(\dot{b}, u) \middle| \dot{b} = f(b,c), u \leq \ell(b,c), c \in A_n\right\}. \tag{41}$$

If $b = b_n$ for some n, we have

$$\hat{F}(b_n) = \{0\} \times \{u \leq \ell(b, y - [r + \delta(1 - q(b_n))]b_n)\}. \tag{42}$$

Otherwise,

$$\hat{F}(b) = \left\{(\dot{b}, u) \middle| \dot{b} \in [f(b, \underline{C}), f(b, \overline{C})], u \leq \ell(b, q(b)(\dot{b} + \delta b) + y - (r+\delta)b)\right\}. \tag{43}$$

Finally, define

$$G(b) \equiv \cap_{\varepsilon > 0} \overline{\text{co}} \left\{(\dot{b}, u) \in \hat{F}(b') \text{ such that } |b' - b| < \varepsilon\right\}. \tag{44}$$

To characterize this set, if $b \neq b_n$, then $G(b) = \hat{F}(b)$ as f is continuous within the open set $\mathcal{M}_n$, $n = 1, ..., N+1$, and the tangent trajectories are generated by $c \in [\underline{C}, \overline{C}]$. For $b = b_n$ for some n, we have

$$G(b_n) = \left\{(\dot{b}, u) | \dot{b} = f(b,c), u \leq \ell(b,c), c \in [\underline{C}, \overline{C}]\right\}.$$

For this case, restricting attention to $c = y - [r + \delta(1 - q(b_n))]b_n$ yields $\hat{F}(b_n)$. Hence BH assumption $\boldsymbol{H2}$ is satisfied.

We use the assumption regarding repurchases around a point of discontinuity in q to rule out the following. Suppose that the following trajectory was feasible: $\dot{b} < -\delta b$ and $c = \liminf_{b \to b_n} q(b_n)(\dot{b} -$

$\delta b) - (r+\delta)b + y > q(b_n)(\dot{b} - \delta b) - (r+\delta)b + y$. Then, in the convexification generating $G(b_n)$, a trajectory featuring $\dot{b} = 0$ and $c > y - [r + \delta(1 - q(b_n))]b$ would appear. This new trajectory would be generated by locating two trajectories featuring $\dot{b} < -\delta b$ and $\dot{b} > -\delta b$, such that their convex combination leads to $\dot{b} = 0$. Because for the trajectory with $\dot{b} > -\delta b$ we have that $c = \overline{C}$, the associated convex combination of the consumptions of these two trajectories would then be strictly greater than the stationary consumption in $\hat{F}(b_n)$, violating $\boldsymbol{H2}$.

BH Proposition 1 and Corollary 1 then imply that the solution to $T\tilde{V}$ is the unique bounded, Lipschitz continuous viscosity solution to

$$\rho(T\tilde{V})(b) = \sup_{c \in [\underline{C}, \overline{C}]} \left\{ c + \lambda D(b|\tilde{V}) + (T\tilde{V})'(b) f(b,c) \right\}.$$

As V is a fixed point of the operator, the government's value V is a viscosity solution to

$$\rho V(b) = H(b, V'(b)) \equiv \sup_{c \in [\underline{C}, \overline{C}]} \left\{ c + \lambda \mathbb{1}_{[V(b) < \overline{V}]} (\overline{V} - V(b)) + V'(b) f(b,c) \right\}, \tag{45}$$

where the term $\lambda \mathbb{1}_{[V(b)<\overline{V}]}(\overline{V} - V(b))$ is taken as a given function of b in verifying the viscosity conditions.

E.5 Proof of Proposition B.2

Proof. We need to verify that if v satisfies the conditions of the proposition, it also satisfies the conditions for a viscosity solution. The proof and details parallel that of the proof for Proposition B.1, and we omit some of the identical steps.

Lower solution conditions. In regard to the conditions for a lower solution, condition (i) in the proposition ensures these are met wherever v is differentiable on the interior of $\boldsymbol{B}$. At the boundaries, $-\overline{a}$ and $\overline{b}$, conditions (iv) and (v) of the proposition state that v equals the stationary value. Hence, $\rho v(b) \leq H(b, \varphi'(b))$, $b \in \{-\overline{a}, \overline{b}\}$, for any $\varphi'(b)$, as $\dot{b} = 0$ is always feasible.

For a non-differentiability at $\underline{b}$, the same argument as for $P(\overline{V})$ in the proof of Proposition B.1 applies. That is, if v has a concave kink, then condition (ii) imposes that value must be the stationary value, which is (weakly) less than $H(b, \varphi'(b))$ for any $\varphi'(b)$. For a convex kink, the lower solution does not impose any restrictions.

At all other points of non-differentiability, condition (iii) states that v has a convex kink, and therefore $v - \varphi$ cannot have a local maximum for a smooth function φ. Thus, the lower solution does not impose any restrictions.

Upper solution conditions. For the upper solution, condition (i) of the proposition states that v satisfies the definition of an upper solution wherever it is differentiable. For points of non-differentiability at $\tilde{b} \neq \underline{b}$, first suppose that q is continuous at $\tilde{b}$. Condition (iii) guarantees that v has a convex kink at $\tilde{b}$, and as in the proof of Proposition B.1, then the convexity of $H(b, \varphi'(b))$ in $\varphi'(b)$ ensures the upper solution inequality is satisfied. If q is not continuous at $\tilde{b}$, then the "tangent trajectories" are restricted to $\dot{b} = 0$. Hence, we need to check that v is weakly greater than the stationary value. This is satisfied by a continuity argument that parallels that in the proof of Proposition B.1.

□

References

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