Online Appendix to

Micro Risks and (Robust) Pareto Improving Policies

A Liquidity Premium on Government Bonds

In this appendix, we provide an alternative perspective on the wedge between the return to physical capital and the interest rate on government bonds. We set $\mu = 1$ and instead appeal to a large body of work documenting that government bonds carry a "convenience yield" or a "liquidity premium," as documented in Krishnamurthy and Vissing-Jorgensen (2012a). In particular, government bonds pay a lower yield than comparable AAA corporate bonds or other non-government safe assets.

We model this as an intermediation technology that uses government debt as an input. Suppose that for every b units of government debt held on its balance sheet, the representative intermediary generates $\rho(\cdot)b$ units of the numeraire good. The arguments of ρ can be any of the aggregate state variables, including the stock of government debt or total output. However, the technology is seen as constant returns to scale from the perspective of an individual competitive intermediary.[48]

The representative intermediary earns $r^k - \delta$ for every unit of capital held and $r^b + \rho(\cdot)$ for every unit of government debt, where r^b is the interest paid on government bonds. Competition in the intermediation sector yields the following arbitrage conditions:

$$r_t^k - \delta = r_t^b + \rho_t = r_t,$$

where r is the interest rate paid to households on deposits.

In what follows, we re-trace the relevant steps of the benchmark analysis. As we proceed, we do not restate the technical assumptions made for each respective result.

A.1 Revisiting Lemma 1 and Corollary 1

As in the benchmark model, the amount raised in any period by the government via factor taxes is

$$F(K_t, N^o) - w^o N^o - (r_t + \delta)K_t,$$

where zero markups imply $\Pi^o = 0$. In the initial equilibrium, the tax revenues are used to pay for the initial debt,

$$r^{bo}B^o = (r^o - \rho^o)B^o = F(K^o, N^o) - w^o N^o - (r^o + \delta)K^o.$$

[48] This intermediation technology for government bonds has antecedents in monetary models, where money is used to reduce transaction costs as in Kimbrough (1986) and Schmitt-Grohé and Uribe (2004). Note, however, that ρ is a function of aggregate variables; for example, the aggregate quantity of government debt. This implies that the "liquidity service" of an individual bond held by an intermediary depends on how large is the total stock of bonds held by the intermediation sector as a whole. In this sense, there is a systemic component to the intermediation technology.

The change in tax revenues for $t \geq 0$ is therefore:

$$F(K^o, N^o) - F(K_t, N^o) - (r_t + \delta)K_t + (r^o + \delta)K^o + (r^o - \rho^o)B^o.$$

The equivalent of (3) is therefore

$$B_{t+1} - (1 + \underbrace{r_t - \rho_t}_{r_t^b})B_t - T_t \geq F(K^o, N^o) - F(K_t, N^o) - (r^o + \delta)K^o + (r_t + \delta)K_t - \underbrace{(r^o - \rho^o)}_{r^{bo}} B^o.$$

The only difference between this expression and (3) is that the rate of government debt $r^b = r - \rho$ differs from the return on capital by ρ_t.

Liquidity services are part of aggregate output (which are included in the interest households earn on deposits). Hence, income accounting implies

$$F(K^o, N^o) + \rho^o B^o = w^o N^o + (r^o + \delta)K^o + r^o B^o.$$

Following the same steps as in the proof of Corollary 1, we obtain

$$C_t \leq F(K_t, N^o) + (1 - \delta)K_t + K_{t+1} + \rho_t B_t.$$

This is the same as in the benchmark, once we recognize liquidity services as part of aggregate output. Note that while increasing government debt generates resources, it may also raise the equilibrium interest rate, requiring the government to intervene in factor markets as in the baseline. This suggests that the elasticity of aggregate savings also plays a role, as shown below.

A.2 Revisiting Corollary 2

Taking the last inequality and subtracting consumption in the initial equilibrium, we obtain

$$\widehat{C}_t \leq F(K_t, N^o) - F(K^o, N^o) + (1 - \delta)\widehat{K}_t - \widehat{K}_{t+1} + (\rho_t B_t - \rho^o B^o),$$

which is the same as in the benchmark given the additional liquidity services.

It is useful to consider a perturbation from a laissez-faire initial equilibrium in which all taxes are zero and $B^o = 0$. This provides a reference that is undistorted by fiscal policy, and hence there are no welfare gains from correcting initial tax distortions. This implies:

$$F_K(K^o, N^o) = r^o + \delta,$$

or $R_k = R^o$, where we recall that $R_k = 1 + F_K(K^o, N^o) - \delta$ and $R^o = 1 + r^o$.

The counterpart of equation (9) is

$$\widehat{K}_{t+1} + \left(\widehat{C}_t - \rho_t B_t\right) \leq R_k \widehat{K}_t.$$

Note that as $B^o = 0$, $\rho_t B_t$ is the change in liquidity services. Hence, the counterpart to equation

(10) is

$$\sum_{t=0}^{\infty} R_k^{-t} \left(\widehat{C}_t - \rho_t B_t\right) \leq 0.$$

This requires that the present value of consumption innovations net of liquidity services is less than zero. Now suppose we have a small innovation to the interest rate at time $\tau > 0$. Following the same steps as in the benchmark analysis, we have

$$\begin{aligned}\sum_{t=0}^{\infty} R_k^{-t} \frac{\partial C_t}{\partial r_\tau} &= (R^o - R_k) \sum_{t=0}^{\infty} R_k^{-t} \frac{\partial \mathcal{A}_t}{\partial r_\tau} + R_k^{-\tau} A^o \\ &= R_k^{-\tau} A^o,\end{aligned}$$

where the second line uses $R^o = R_k$, as there is no markup.

Assuming regularity conditions for ρ and small changes to B_t, we can approximate

$$\rho_t B_t \approx \rho^o (B_t - B^o) + (\rho_t - \rho^o) B^o = \rho^o B_t,$$

where the last equality uses $B^o = 0$ and ρ^o is the marginal product of liquidity services in the initial equilibrium.

At the margin, the returns to physical capital net of depreciation and to government bonds inclusive of liquidity services are equated in the initial equilibrium. To a first order, it therefore is irrelevant whether changes in household wealth are backed by changes in K_t or B_t. For expositional purposes, suppose changes in household wealth are equivalent to changes in government bonds

$$\frac{B_t - \cancelto{0}{B^o}}{\Delta r_\tau} = \frac{\Delta \mathcal{A}_t}{\Delta r_\tau}$$

and therefore for small changes we have[49]

$$\rho_t B_t \approx \rho^o \frac{\partial \mathcal{A}_t}{\partial r_\tau} \times \Delta r_\tau.$$

The sufficient condition for a feasible RPI becomes

$$R_k^{-\tau} A^o - \rho^o \sum_{t=0}^{\infty} R_k^{-t} \frac{\partial \mathcal{A}_t}{\partial r_\tau} < 0.$$

Rearranging, and using our definition of $\xi_{t,\tau}$ from the benchmark, the counterpart of equation (14) becomes

$$\frac{\rho^o}{R^o} \sum_{t=0}^{\infty} R_k^{-(t-\tau)} \xi_{t,\tau} > 1.$$

[49]If $B^o \neq 0$, the ρ^o in the following expression would be replaced by $\rho^o \left(1 + \frac{\partial\rho/\partial B \times B^o}{\rho^o}\right)$. The latter term is the elasticity of the convenience yield to changes in government bonds. It is this elasticity that is the focus of event studies surrounding quantitative easing (QE) episodes, such as Krishnamurthy and Vissing-Jorgensen (2012b) and Koijen, Koulischer, Nguyen, and Yogo (2021).

This is similar to the benchmark's equation (14), but with the liquidity premium replacing the wedge between the marginal product of capital and the interest rate. In the benchmark, the government could exploit that wedge, which existed because of a markup. In this alternative, the government can generate liquidity services by issuing debt. The larger the marginal product of bonds in generating liquidity services, the easier it is to satisfy feasibility.[50] We obtain the result that the roles of R_k and $\xi_{t,\tau}$ in the infinite sum is exactly the same as in the benchmark.

B Transfers When Capital is Below the Golden Rule

Consider the following notion of monotonicity of aggregate consumption with respect to transfers:

Definition 5. *We say $\boldsymbol{C} = \{C_t\}_{t\geq 0}$ is weakly increasing in $\boldsymbol{T}$ if $\boldsymbol{T}' \geq \boldsymbol{T}$ implies $\boldsymbol{C}(\boldsymbol{r}, \boldsymbol{T}') \geq \boldsymbol{C}(\boldsymbol{r}, \boldsymbol{T})$, where the inequality holds for all t in the respective sequences. If $\boldsymbol{T}' \geq \boldsymbol{T}$ for which there is a t such that $T_t' > T_t$ implies $\boldsymbol{C}(\boldsymbol{r}, \boldsymbol{T}') \geq \boldsymbol{C}(\boldsymbol{r}, \boldsymbol{T})$ and that there is an s such that $\boldsymbol{C}_s(\boldsymbol{r}, \boldsymbol{T}') > \boldsymbol{C}_s(\boldsymbol{r}, \boldsymbol{T})$, we say $\boldsymbol{C}$ is strictly increasing in $\boldsymbol{T}$.*

This is a natural property, in that holding constant all interest rates, one would naturally expect an increase in lump sum transfers would induce households (in aggregate) to consume more.[51]

The following result says that if consumption is weakly increasing in transfers, then we can ignore the role of transfers when looking for an RPI (as long as an interest rate have changed). That is, transfers are not *necessary* for evaluating feasibility:

Lemma 5 (Transfers are not necessary)**.** *Suppose that $\boldsymbol{C}$ is weakly increasing in $\boldsymbol{T}$. Let $(\boldsymbol{r}, \boldsymbol{T})$ be a feasible RPI where for some t, $r_t > r^o$. Then $(\boldsymbol{r}, \boldsymbol{T}')$ where $\boldsymbol{T}' = \{-(r_t - r^o)\underline{a}\}_{t\geq 0}$ is also a feasible RPI.*

Proof. Note that an RPI requires that $r_t \geq r^o$ and $T_t \geq -(r_t - r^o)\underline{a}$. The fact that the $\boldsymbol{C}$ is weakly increasing implies that $\boldsymbol{C}_t(\boldsymbol{r}, \boldsymbol{T}) \geq \boldsymbol{C}_t(\boldsymbol{r}, \boldsymbol{T}')$, as $\boldsymbol{T} \geq \boldsymbol{T}'$. The sequence of K_t that implements the $(\boldsymbol{r}, \boldsymbol{T})$ then also implements $(\boldsymbol{r}, \boldsymbol{T}')$. Given that $r_t > r^o$ for some t, it follows that $(\boldsymbol{r}, \boldsymbol{T}')$ is a feasible RPI. □

The following result says that if consumption is strictly increasing in transfers, than an RPI is not feasible without a change in an the interest rate. That is, transfers alone are not *sufficient*:

Lemma 6 (Transfer are not sufficient)**.** *Suppose that $\boldsymbol{C}$ is strictly increasing in $\boldsymbol{T}$. If $K^o < K^\star$, then there is no feasible RPI in which $\boldsymbol{r} = \boldsymbol{r}^o$.*

[50] If Ricardian equivalence held, then a version of the Friedman rule would apply; that is, the government should issue debt until the marginal return to liquidity services is driven to zero. Here, issuing debt is not neutral, and hence will change allocations and factor prices and potentially violate the requirements of an RPI.

[51] For an individual agent in incomplete markets it is possible to construct examples where individual consumption falls given an increase in future transfers. However, we are counting on heterogeneity to guarantee that such individual behavior does not aggregate. Wolf (2021) presents examples of permanent income and hand to mouth households where these assumptions hold. See also Farhi, Olivi, and Werning, 2022 for general comparative statics results for incomplete market economies.

Proof. Suppose there is a feasible RPI, $(\boldsymbol{r}^o, \boldsymbol{T})$. There must be a non-negative sequence of $\{K_t\}_{t=0}^{\infty}$ such that

$$\boldsymbol{C}_t(\boldsymbol{r}^o, \boldsymbol{T}') + K_{t+1} \leq F(K_t, N^o) + (1-\delta)K_t$$

Exploiting the concavity of technology, and that $C^o = F(K^o, N^o) - \delta K^o$, we have that

$$K_{t+1} - K^o \leq (R_k - 1)(K_t - K^o) - (\boldsymbol{C}_t(\boldsymbol{r}^o, \boldsymbol{T}) - C^o).$$

Note that $R_k > 1$, together with $\boldsymbol{C}$ increasing in $\boldsymbol{T}$, implies that $K_t \leq K^o$ for all t.
Let s be the first time where $\boldsymbol{C}_{s-1}(\boldsymbol{r}^o, \boldsymbol{T}) > C^o$ (such a time exists, given that $\boldsymbol{C}$ is strict increasing in $\boldsymbol{T}$). Then, the above implies that $K_s < K^o$. Now note that

$$K_{s+m} - K^o \leq (R_k - 1)^m (K_s - K^o)$$

Given that $R_k > 1$, it follows then that $K_t < 0$ for t large enough, a contradiction. □

When we focus on the case where the economy operates below the Golden Rule, the above result tells us that in a feasible RPI (under a reasonable assumption on $\boldsymbol{C}$) an interest rate must changes at some date. The reason is that with only increases in transfers, aggregate consumption will be higher at all times with the RPI than originally, an impossibility given the resource constraint and $K^o < K^\star$.

C Proofs

C.1 Proof of Lemma 1

Towards sufficiency, suppose that the conditions of the lemma hold. Then, for $t \geq 0$, set τ_t^n such that

$$\frac{F_N(K_t, N^o)}{(1+\tau_t^n)\mu} = w^o.$$

This ensures the labor market clears at $w_t = w^o$ and $N_t = N^o$, where GHH preferences ensure that the households are willing to supply N^o at wage w^o. Note that as $K_0 = K^o$ is given, τ_0^n is the same as the initial equilibrium. Similarly, the government taxes or subsidizes profits so that

$$\Pi_t = (1-\tau_t^\pi)\widetilde{\Pi}_t = (1-\tau_t^\pi)(\mu - 1)F(K_t, N^o)/\mu = \Pi^o.$$

This determines τ_t^π. Given that at $t = 0$, K_0 is given, τ_0^π is as in the initial equilibrium.

Finally, the government must ensure that the representative firm's choice of capital is consistent with the risk-free interest rate for all $t \geq 1$:

$$F_K(K_t, N^o) = (1+\tau_t^k)\mu r_t^k = (1+\tau_t^k)\mu(r_t + \delta),$$

which then determines τ_t^k. For $t = 0$, as highlighted in footnote 20, we require that r_0^k remains unchanged, and as K_0 is given and $r_0 = r^o$, τ_0^k remains as in the original equilibrium.

The sequence of tax rates defined above ensure that firms optimize and markets clear for labor and capital. By definition of $\mathcal{A}_t$ and condition (i) of the lemma, the market for assets also clears given $\{r_t, T_t\}$.

The final equilibrium condition involves government revenues and transfers. The total government revenue (before transfers) of this tax policy at all $t \geq 0$ is given by

$$\begin{aligned}
\text{Revenue} &= \tau_t^n w^o N^o + \tau_t^k r_t^k K_t + \tau_t^\pi \widetilde{\Pi}_t \\
&= (1 + \tau_t^n) w^o N^o + (1 + \tau_t^k) r_t^k K_t - (1 - \tau_t^\pi)\widetilde{\Pi}_t - w^o N^o - r_t^k K_t + \widehat{\Pi}_t \\
&= \frac{F_N(K_t, N^o) N^o + F_K(K_t, N^o) K_t}{\mu} - \Pi^o - w^o N^o - r_t^k K_t + \frac{(\mu - 1) F(K_t, N^o)}{\mu} \\
&= F(K_t, N^o) - \Pi^o - w^o N^o - r_t^k K_t,
\end{aligned}$$

where the third line uses $(1 - \tau_t^\pi)\widehat{\Pi}_t = \Pi^o$; the firm's first-order condition for labor and capital; and $\widehat{\Pi}_t = (\mu - 1)F/\mu$. The last line follows from Euler's theorem. Note that national income accounting implies

$$F(K^o, N^o) = \Pi^o + w^o N^o + r^{ko} K^o + r^o B^o.$$

Hence, we can replace $\Pi^o + w^o N^o = F(K^o, N^o) - r^{ko} K^o - r^o B^o$ and $r_t^k = r_t + \delta$ to obtain

$$\text{Revenue} = F(K_t, N^o) - F(K^o, N^o) - (r_t + \delta) K_t + (r^o + \delta) K^o + r^o B^o. \tag{C.17}$$

As transfers equals revenue plus net debt issuance, we have

$$T_t \leq F(K_t, N^o) - F(K^o, N^o) - (r_t + \delta) K_t + (r^o + \delta) K^o + r^o B^o + B_{t+1} - (1 + r_t) B_t,$$

where the inequality allows for free disposal of government surpluses. This is condition (3), and thus ensures that the government has a non-negative surplus at every t given the proposed taxes, transfers, and debt issuances. This establishes that given the sequences in the premise, we can construct a tax plan that implements an equilibrium.

Necessity of condition (i) in the lemma follows from the market clearing condition in the definition of equilibrium. The necessity of condition (ii) follows from firm optimization and the government budget constraint. ◻

C.2 Proof of Corollary 1

Using

$$F(K^o, N^o) = w^o N^o + \Pi^o + (r^o + \delta) K^o + r^o B^o,$$

we have

$$C_t = F(K^o, N^o) - (r^o + \delta) K^o - r^o B^o + (1 + r_t)\mathcal{A}_t - \mathcal{A}_{t+1} + T_t.$$

Using $\mathcal{A}_t = K_t + B_t$, this is equivalent to

$$C_t + K_{t+1} = F(K^o, N^o) - (r^o + \delta) K^o - r^o B^o + (1 + r_t)(K_t + B_t) - B_{t+1} + T_t.$$

Substituting into (6) and re-arranging gives (3). ◻

C.3 Proof of Lemma 2

For a given ν, let $\boldsymbol{T}' = \boldsymbol{T}^o + \widehat{T}\nu$ be the new transfer sequence. From the continuity condition, we have

$$C_t(\boldsymbol{r}^o, \boldsymbol{T}') - C^o \leq |C_t(\boldsymbol{r}^o, \boldsymbol{T}') - C^o| \leq M\nu.$$

For $t = 0$, we have

$$\begin{aligned}
C_t(\boldsymbol{r}^o, \boldsymbol{T}') + K^\star &\leq C^o + M\nu + K^\star \\
&= F(K^o, N^o) - \delta K^o + K^\star + M\nu \\
&= F(K^o, N^o) + (1-\delta)K^o + (M\nu + K^\star - K^o)
\end{aligned}$$

and hence the condition in Corollary 1 holds for $0 < \nu \leq (K^o - K^\star)/M \equiv \nu_1$, as $K^o > K^\star$.

For $t \geq 1$, it is sufficient if

$$M\nu + C^o \leq F(K^\star, N^o) - \delta K^\star,$$

or, using $C^o = F(K^o, N^o) - \delta K^o$,

$$M\nu \leq F(K^\star, N^o) - \delta K^\star - (F(K^o, N^o) - \delta K^o)\,.$$

Letting $\nu_2 \equiv M^{-1}\left(F(K^\star, N^o) - \delta K^\star - (F(K^o, N^o) - \delta K^o)\right) > 0$, this condition is satisfied if $0 < \nu \leq \nu_2$.

Collecting, for $0 < \nu \leq \min\{\nu_1, \nu_2\}$, the transfer scheme $\boldsymbol{T}' = \boldsymbol{T}^o + \widehat{\boldsymbol{T}}\nu$ is implementability and represents an RPI. □

C.4 Proof of Proposition 1

For a given ν, let $\boldsymbol{r}' \equiv \boldsymbol{r}^o + \widehat{\boldsymbol{r}}\nu$. Note that $(\boldsymbol{r}', \boldsymbol{T}^o)$ is an RPI. Let us propose the following sequence of $\{K_t\}_{t=0}^\infty$:

$$K_t = K^o + R_k^{-1}\sum_{s=0}^{\infty} R_k^{-s}(C_{t+s}(\boldsymbol{r}', \boldsymbol{T}^o) - C^o) + h\nu, \qquad \text{for } t \geq 1.$$

with $K_0 = K^o$. We will check that such sequence implements $(\boldsymbol{r}', \boldsymbol{T}^o)$ for ν small enough.

Note that

$$\begin{aligned}
|K_t - K^o| &\leq R_k^{-1}\sum_{s=0}^{\infty} R_k^{-s}|C_{t+s}(\boldsymbol{r}', \boldsymbol{T}^o) - C^o| + h\nu \\
&\leq R_k^{-1}\sum_{s=0}^{\infty} R_k^{-s}\nu M + h\nu = \left[\left(\frac{1}{R_k - 1}\right)M + h\right]\nu \equiv M_0\nu,
\end{aligned}$$

where the second line uses property (ii). Then, there exists $\nu_1 < \epsilon$ such that $K_t > 0$ for all $t \geq 0$ and $\nu < \nu_1$.

Let

$$\overline{F}_{KK} \equiv -\sup_K\{|F_{KK}(K, N^o)|\colon |K - K^o| \leq M_0 \nu_1\}.$$

As F_{KK} is continuous and this is a compact domain, $\overline{F}_{KK}$ is finite. Note that for $\nu < \nu_1$, Taylor's theorem implies that

$$F(K_t, N^o) + (1-\delta)K_t = F(K^o, N^o) + (1-\delta)K^o + R_k(K_t - K^o) + \frac{1}{2}F_{KK}(\widetilde{K}, N^o)(K_t - K^o)^2$$

for some $\widetilde{K}$ between K^o ad K_t. Using that $F(K^o, N^o) + (1-\delta)K^o = C^o + K^o$ and that $|K_t - K^o| \leq M_0\nu$, we have that

$$F(K_t, N^o) + (1-\delta)K_t \geq C^o + K^o + R_k(K_t - K^o) + \frac{\overline{F}_{KK}}{2}(M_0\nu)^2$$

Then, a sufficient condition for (6) from Corollary 1 is

$$\begin{aligned}
C_0(\boldsymbol{r}', \boldsymbol{T}^o) + K_1 &\leq C^o + K^o \\
C_t(\boldsymbol{r}', \boldsymbol{T}^o) + K_{t+1} &\leq C^o + K^o + R_k(K_t - K^o) + \frac{\overline{F}_{KK}}{2}(M_0\nu)^2, \text{ for all } t \geq 1.
\end{aligned}$$

For the first inequality, using the proposed K_1, we have that

$$\sum_{s=0}^{\infty} R_k^{-s}(C_s(\boldsymbol{r}', \boldsymbol{T}^o) - C^o) + h\nu \leq 0$$

which holds given (i).

For the second inequalities, using the proposed $\{K_t\}$, we have

$$\begin{aligned}
\sum_{s=0}^{\infty} R_k^{-s}(C_{t+s}(\boldsymbol{r}', \boldsymbol{T}^o) - C^o) + h\nu &\leq \sum_{s=0}^{\infty} R_k^{-s}(C_{t+s}(\boldsymbol{r}', \boldsymbol{T}^o) - C^o) + R_k h\nu + \frac{\overline{F}_{KK}}{2}(M_0\nu)^2 \\
0 &\leq (R_k - 1)h\nu + \frac{\overline{F}_{KK}}{2}(M_0\nu)^2
\end{aligned}$$

Given that $h > 0$, there exists $\nu_2 > 0$ such that

$$(R_k - 1)h \geq -\frac{\overline{F}_{KK}}{2}M_0^2\nu$$

for all $\nu \in (0, \nu_2)$.

Let $\overline{\nu} = \min\{\nu_1, \nu_2\}$. Then $(\boldsymbol{r}', \boldsymbol{T}^o)$ for any $\nu \in (0, \overline{\nu})$ is a feasible RPI. □

C.5 Proof of Corollary 2

Divide both sides of equation 12 by $R_K^{-\tau}A^o$, factor out R^o, and use the definition of $\xi_{t,\tau}$ to obtain (14). As shown in the text, this implies (11) is satisfied, which in turn is sufficient for (i) in Proposition 1. Condition (ii) holds by the differentiability of C_t, which is implied by the differentiability of $\mathcal{A}_t$

stated in the premise. ◻

C.6 Proof of Lemma 3

As in the benchmark model's Corollary 2, consider a policy that sets $r_t = r^o$ for all $t \neq \tau$ and $r_\tau = r^o + \Delta r_\tau$ for some $\tau > 0$ and $\Delta r_\tau > 0$. Recall that in the representative agent environment, $R^o \equiv 1 + r^o = 1/\beta$. From the Euler equation, we have

$$c_t = \begin{cases} \underline{c} \text{ for } t \leq \tau - 1 \\ \overline{c} \text{ for } t \geq \tau, \end{cases}$$

where $\underline{c}$ and $\overline{c}$ satisfy the Euler equation at time $\tau - 1$:

$$u'(\underline{c}) = \beta(1 + r_\tau)u'(\overline{c}).$$

For small changes around the initial equilibrium consumption C^o, we can differentiate this to obtain:

$$u''(C^o)\frac{d\underline{c}}{dr_\tau} = \beta u'(C^o) + u''(C^o)\frac{d\overline{c}}{dr_\tau},$$

where we use the fact that $1 + r^o = 1/\beta$. Rearranging, we have

$$\frac{d\overline{c}}{dr_\tau} - \frac{d\underline{c}}{dr_\tau} = C^o \beta \zeta, \tag{C.18}$$

where $\zeta = -u'(C^o)/(u''(C^o)C^o)$.

Using $\beta = 1/R^o$, the budget constraint requires:

$$\underline{c}\sum_{t=0}^{\tau-1}\beta^t + \frac{\overline{c}}{1 + r_\tau}\sum_{t=0}^{\infty}\beta^{t+\tau-1} = R^o A^o + (w^o N^o + \Pi^o)\left(\sum_{t=0}^{\tau-1}\beta^t + \frac{1}{1 + r_\tau}\sum_{t=0}^{\infty}\beta^{t+\tau-1}\right).$$

Differentiating and using $C^o = w^o N^o + \Pi^o + r^o A^o$, we obtain:

$$\frac{d\underline{c}}{dr_\tau} + \beta^\tau\left(\frac{d\overline{c}}{dr_\tau} - \frac{d\underline{c}}{dr_\tau}\right) = \beta^{\tau+1} r^o A^o.$$

Combining this with (C.18), we obtain:

$$\begin{aligned} \frac{d\underline{c}}{dr_\tau} &= \beta^{\tau+1}\left(r^o A^o - \zeta C^o\right) \\ \frac{d\overline{c}}{dr_\tau} &= \frac{d\underline{c}}{dr_\tau} + \beta\zeta C^o. \end{aligned}$$

This implies

$$
\begin{aligned}
\sum_{t=0}^{\infty} R_k^{-t} \frac{\partial C_t}{\partial r_\tau} &= \sum_{t=0}^{\infty} R_k^{-t} \frac{d\underline{c}}{dr_\tau} + \sum_{t=\tau}^{\infty} R_k^{-t} \left(\frac{d\bar{c}}{dr_\tau} - \frac{d\underline{c}}{dr_\tau} \right) \\
&= \left(\frac{1}{1 - R_k^{-1}} \right) \beta^{\tau+1} \left(r^o A^o - \zeta C^o \right) + \left(\frac{R_k^{-\tau}}{1 - R_k^{-1}} \right) \beta \zeta C^o .
\end{aligned}
$$

Letting $\tau \longrightarrow \infty$, equation (11) is satisfied if $r^o A^o < \zeta C^o$, or $\zeta > \frac{r^o A^o}{C^o}$, which is the condition in the lemma. □

C.7 Proof of Proposition 2

Towards a contradiction, suppose there is a feasible RPI, $(\boldsymbol{r}, \boldsymbol{T})$. Given that we start from the laissez-faire allocation, this requires that $\boldsymbol{T}$ is non-negative. From the feasibility condition in Corollary 1, we have that there exists a sequence of K_t such that

$$
\begin{aligned}
w^o N^o + \Pi^o + (1 + r_t)\mathcal{A}(\boldsymbol{r}, \boldsymbol{T}) - \mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) + T_t &\leq F(K_t, N^o) + (1 - \delta)K_t - K_{t+1} \\
&\leq \underbrace{F(K^o, N^o) + (1 - \delta)K^o}_{C^o + K^o} + R_k(K_t - K^o) - K_{t+1}
\end{aligned}
$$

where the last inequality follows from concavity of F. Using that $R_k = 1 + r^o$ as $\mu = 1$, we have

$$
\begin{aligned}
w^o N^o + \Pi^o + T_t + (1 + r_t)\mathcal{A}_t(\boldsymbol{r}, \boldsymbol{T}) - \mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) &\leq C^o + (1 + r^o)(K_t - K^o) - (K_{t+1} - K^o) \\
-r^o A^o + (1 + r_t)\mathcal{A}_t(\boldsymbol{r}, \boldsymbol{T}) - \mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) + T_t &\leq (1 + r^o)(K_t - K^o) - (K_{t+1} - K^o)
\end{aligned}
$$

And thus

$$
\mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) - K_{t+1} \geq (1 + r_t)(\mathcal{A}_t(\boldsymbol{r}, \boldsymbol{T}) - K_t) + (r_t - r^o)K_t + T_t
$$

Note that starting from the laissez-faire implies that $K^o = A^o$, and thus $\mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) - K_{t+1}$ is always non-negative, and turns strictly positive whenever $r_t > r^o$ or $T_t > 0$. Hence, we have that

$$
\mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) - K_{t+1} \geq (1 + r^o)(\mathcal{A}_t(\boldsymbol{r}, \boldsymbol{T}) - K_t) + (r_t - r^o)K_t + T_t
$$

and given that $r^o > 0$ ($K^o < K^\star$ and $\mu = 1$), it follows that $\mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) - K_{t+1}$ must necessarily go to infinity at t increases. The finite technology implies that K_t must remain bounded, and thus $\mathcal{A}_{t+1}(\boldsymbol{r}, \boldsymbol{T}) \longrightarrow \infty$. The assumption in the proposition then implies that for any M there exists a s such that $C_s(\boldsymbol{r}, \boldsymbol{T}) > M$. For M sufficiently large, the resource constraint at s must be violated, generating the contradiction. □

C.8 Proof of Lemma 4

The proof follows similar steps as the proof of Corollary 1. Factor taxes are defined in the same manner as in the proof of that corollary. The government budget constraint is:

$$T_t(s^t) \leq F(s^t, K_t(s^{t-1}), N_t^o(s^t)) - F(s^t, K_t^o(s^{t-1}), N_t^o(s^t)) - \\ r_t^k(s^t)K_t(s^{t-1}) + r_t^{ko}(s^t)K_t^o(s^{t-1}) + B_{t+1}(s^t) - (1 + r_t(s^{t-1}))B_t(s^{t-1}).$$

The aggregated household budget set is:

$$C_t(s^t) = w_t^o(s^t)N_t^o(s^t) + \Pi_t^o(s^t) + (1 + r_t^k(s^t) - \delta)K_t(s^{t-1}) - K_{t+1}(s^t) + (1 + r_t(s^{t-1}))B_t(s^{t-1}) - B_{t+1}(s^t) + T_t(s^t).$$

We have

$$F(s^t, K_t^o(s^{t-1}), N_t^o(s^t)) = w_t^o(s^t)N_t^o(s^t) + \Pi_t^o(s^t) + r_t^{ko}(s^t)K_t^o(s^{t-1}).$$

Using this to substitute for $w_t^o(s^t)N_t^o(s^t) + \Pi_t^o(s^t)$ in the HH budget constraint, and then use the resulting expression to substitute for T_t in the government budget constraint, we obtain the expression in the lemma:

$$C_t(s^t) \leq F(s^t, K_t(s^{t-1}), N_t^o(s^t)) + (1 - \delta)K_t(s^{t-1}) - K_{t+1}(s^t).$$

This condition ensures that the government budget constraint and aggregate market clearing hold, given a sequence of functions $C_t(s^t)$ and $K_{t+1}(s^t)$. A necessary and sufficient condition for equilibrium is that the aggregate household policy for consumption, $C(s^t; \boldsymbol{r}, \boldsymbol{r}\boldsymbol{T})$ satisfies the above resource condition and the sequence $K(s^t) \in \mathcal{K}(s^t; \boldsymbol{r}, \boldsymbol{r}\boldsymbol{T})$. □

D Simulation

D.1 Preferences and Technology

The utility function we consider for households is of the Epstein-Zin form

$$V_{it} = \left\{ (1 - \beta)x_{it}^{1-1/\zeta} + \beta \left(\mathbb{E}_z V_{it+1}^{1-\gamma} \right)^{\frac{1-1/\zeta}{1-\gamma}} \right\}^{\frac{1}{1-1/\zeta}}, \tag{D.19}$$

where β is the discount factor, ζ is the elasticity of intertemporal substitution, γ is the risk aversion coefficient, and x is the composite of consumption and labor $x_{it} = c_{it} - n_{it}^{1/\nu}$. The parameter ν controls the Frisch elasticity of the labor supply. We set some of the preference parameters to conventional values in the literature and others as part of the calibration. The elasticities of intertemporal substitution and of labor supply are set to the common parameter values of 1 and 0.2, respectively. The discount factor and coefficient of risk aversion are set as part of the calibration exercise described below. We set the borrowing constraint to zero for all households.

An important part of the parametrization is the stochastic structure for idiosyncratic shocks. We adopt the structure and estimates from Krueger, Mitman, and Perri (2016), which use micro data on after-tax labor earnings from the PSID. Idiosyncratic productivity shocks z_{it} contain a persistent and a transitory component, and their process is as follows: $\log z_{it} = \tilde{z}_{it} + \varepsilon_{it}$ and

$\tilde{z}_{it} = \rho^z \tilde{z}_{it-1} + \eta_{it}$, with persistence ρ^z and innovations of the persistent and transitory shocks (η, ε), and associated variances given by $(\sigma^2_\eta, \sigma^2_\varepsilon)$. We set the three parameters controlling this process $(\rho^z, \sigma^2_\eta, \sigma^2_\varepsilon)$ to .9695, .0320, and .0435, respectively, to reflect the estimated earnings risk in Krueger, Mitman, and Perri (2016) for employed individuals and the endogenous labor supply decision in our model. We discretize this process into 10 points, based on the Rouwenhurst method.

As mentioned in the text, we take a parsimonious approach to allocating profits to households and assume a distinct class of entrepreneurs who are endowed with managerial talent and consume profit distributions in a hand-to-mouth manner.

The technology specification is Cobb-Douglas, $F(K, N) = K^\alpha N^{1-\alpha}$. We use standard values for the coefficient α and for the depreciation rate of capital δ. The values are $\alpha = 0.3$ and $\delta = 0.1$. The markup parameter μ is set to 1.4.

We calibrate the discount factor and the coefficient of relative risk aversion as follows. We target a steady state with 60% debt-to-output and capital-to-output of 2.5, where the debt corresponds to the US average over the period 1966-2021 and the capital ratio is taken from Aiyagari and McGrattan (1998). We treat this steady state as the result of a constant-K policy starting from a laissez-faire economy. The average interest rate relative to growth in the US over the sample period is -1.4%, which will be the target for the return on bonds in our steady state. The resulting parameter values are a discount factor of $\beta = 0.993$ and a coefficient of risk avers is $\gamma = 5.5$.

D.2 Constant-*K* Simulation

Our "baseline fiscal policy" is the one which keeps capital constant starting from the laissez-faire.

Table 2: Baseline Constant-*K* Policy and Laissez-Faire Economies

	Data	Constant-*K* Policy	Laissez-Faire
Aggregates			
Public Debt (% output)	60	60	0
Interest Rates(%)	-1.4	-1.4	-1.7
Capital (rel. output)	2.5	2.6	2.6
Wealth Distribution			
Q1 Wealth Share	-1	1	1
Q2 Wealth Share	1	4	4
Q3 Wealth Share	4	11	10
Q4 Wealth Share	13	23	23
Q5 Wealth Share	83	61	63

Table 2 presents some moments in the stationary equilibrium of the economy with baseline constant-*K* fiscal policy and the laissez-faire economy. The levels of public debt, interest rates, and capital in the economy with the baseline fiscal policy match the data moments by construction.[52] The table shows that an increase in debt to output of 60% raises interest rates by 0.3 percentage points. We also present some moments on the wealth distribution in the steady states—namely the wealth share of each asset quintile—and compare them with data as reported in Krueger,

[52]The economy is dynamically efficient, also by construction. To see this, $F_K = \alpha Y/K = 0.3/2.5 = 0.12$, which is greater than the depreciation rate of 0.10.

Mitman, and Perri (2016). Our model economies generate skewed distributions of wealth, with most of the wealth being held by the top quintile of the distribution, although they are not quite as skewed as the data. In addition in our model economies, a small fraction of agents, about 2%, are at their borrowing constraint at any period.

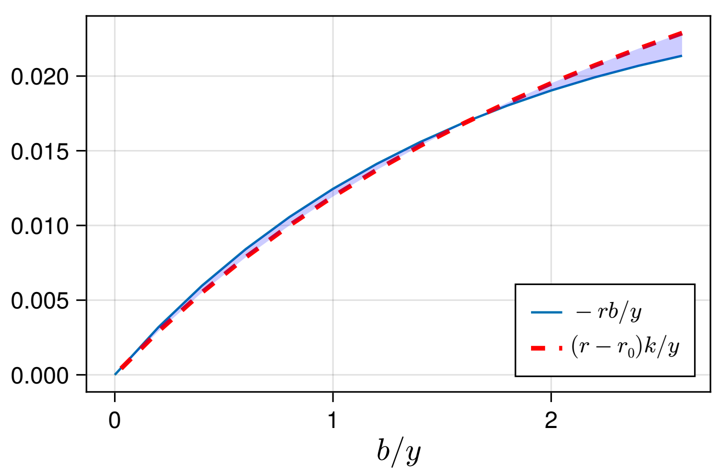
D.3 Debt Laffer Curve

We revisit the logic of Figure 1. In particular, long-run seigniorage is given by $-rB$, while the costs are captured by $\Delta r \times K_0$. In Figure D.1, we plot these two components for *stationary equilibria* with different levels of debt to output for the constant-K policy studied in subsection 5.2. At each debt level, tax policy is set to deliver laissez-faire wages and profits. As can be seen, up until debt levels of roughly 1.7 times the level of output, seigniorage exceeds fiscal costs, implying positive lump-sum transfers to households. Beyond this level of debt, the increase in interest rates makes weakly positive transfers infeasible.

Note that these two curves intersect while seigniorage is still increasing in debt. Eventually, r becomes close enough to zero that seigniorage begins to decline in debt. The peak of this Laffer curve occurs at debt levels roughly four times output. Feasible Pareto-improving levels of debt consistent with a constant-K policy, however, are much lower than this peak.

While Figure D.1 establishes only that the policy is feasible in the new steady state, the analysis of transition dynamics in the baseline case above suggests that feasibility in the steady state is the critical metric. Along the transition, the government is a net issuer of bonds. As long as the revenue from the net issuances dominates any overshooting of the interest rate, feasibility rests on long-run considerations.

Figure D.1: Steady-State Seigniorage and Tax Revenue across Debt



D.4 Computational Algorithm

This appendix describes the computational algorithm we use in solving the model. The code is available at

https://github.com/manuelamador/micro_risks_pareto_improving_policies.

Our procedure consists of three steps. First, we compute the initial and final stationary equilibria. The initial one is the laissez-faire equilibrium and the final one has fiscal policy active. A second step computes the transition of this economy. Finally, we compute the aggregate savings elasticities associated with an initial laissez-faire equilibrium and operationalize Corollary 2.

D.5 Stationary Equilibrium

The computations of the policy and value functions rely on an endogenous grid method, modified for the presence of Epstein-Zin preferences. In particular, we use the value function, equation (D.19), together with the first order condition with respect to consumption:

$$(1-\beta)x_{it}^{-1/\zeta} \geq \beta \left(\mathbb{E}_z V_{it+1}^{1-\gamma}\right)^{\frac{\gamma-1/\zeta}{1-\gamma}} \mathbb{E}_z \left(V_{it+1}^{-\gamma} \frac{dV_{it+1}}{a_{it+1}}\right).$$

The envelope condition implies

$$\frac{dV_{it+1}}{a_{it+1}} = (1-\beta)R_{t+1}V_{it+1}^{1/\zeta}x_{it+1}^{-1/\zeta}.$$

Taken together, we obtain the following Euler equation:

$$x_{it}^{-1/\zeta} \geq \beta \left(\mathbb{E}_z V_{it+1}^{1-\gamma}\right)^{\frac{\gamma-1/\zeta}{1-\gamma}} \mathbb{E}_z \left(V_{it+1}^{1/\zeta-\gamma} R_{t+1} x_{it+1}^{-1/\zeta}\right) \tag{D.20}$$

We let $\eta_{it} \equiv R_t^{-\zeta} x_{it}$.

Initial. To compute the initial laissez-faire stationary equlibrium, we proceed as follows. Given a guess for the initial interest rate R^o, we obtain the wage level consistent with the technology w^o. We then solve the household problem given wages and interest rates, w^o, R^o (and set $T^o = 0$). We do this as follows. Given the wage, the labor supply is easily obtained from the GHH preferences. We then iterate *backwards* using an endogenous grid method based on (D.20) and the value function (D.19). That is, we start with a guess for V_{it+1} and η_{it+1} and use the Euler equation and the value function to compute the values of V_{it} and η_{it} that are consistent with the guess and the borrowing constraint, using a linear interpolation. We iterate until V and η have converged to some tolerance.

Having solved the households problem, we use the stationary policy function to obtain a transition function for the distribution of households (as in Young, 2010), and compute the implied stationary distribution, $\Delta^o(a,z)$. To obtain the stationary general equilibrium, we repeat this for different values of R^o until the aggregate of household savings in the stationary state is consistent (for a given tolerance) with the capital stock given R^o and the implied total labor supply, N^o.

Final. The final stationary equilibrium computation follows a similar approach as the initial one. In this case, we know that the wage, and the labor supply remain equal to the values in the initial equilibrium. For a given guess of the interest rate R^1, a target level of government debt B^1 and a long-run level of capital K^1, we use the government budget constraint to obtain the implied transfers, T^1, that make the government budget constraint hold with equality in the stationary equilibrium (using inequality (3) with equality). We then solve the household problem given w^o, R^1 and T^1. As in the initial stationary equilibrium, we iterate on R^1 (and obtaining a new T^1) until the aggregate of the household savings equal the sum of K^1 and B^1.

D.6 Transition

At time 0, the government announces a sequence of fiscal policies that implements a sequence of capital and debt $\{K_t, B_t\}_{t=0}^{H}$. We will assume that at period H, the economy is in the final stationary equilibrium, with $K_H = K^1$ and $B_H = B^1$.

We use Lemma 1 to compute the transition as follows. We start with a guess of interest rates $\{R_0, R_1, R_2, ..., R_H\}$ with $R_0 = R^o$ and $R_H = R^1$. Given this guess, we can use (3) with equality to obtain the sequence of implied transfers, T_t. Starting from the value function V^1 and the additional state η^1 set at the values of the final equilibrium, we use the Euler equation and the value function to iterate backwards and construct a sequence of V_t and η_t. With these sequences, we compute the policy functions and the transition function for the distribution of households. We then, starting from Λ^o, iterate forward the evolution of the distribution. With this, we compute the aggregate of the household savings at each time, A_t. We then look for a root: a sequence $\{R_t\}$ such that $A_t = B_t + K_t$ for all $t \leq H$ (up to some tolerance), as required by Lemma 1.[53]

D.7 Transition with Aggregate Shocks

We extend our algorithm to incorporate aggregate uncertainty that is resolved in period 1. We start the economy at time $t = 0$ from the same initial stationary laissez-faire equilibrium as in the previous examples. Agents understand that at $t = 1$ the economy is hit with an aggregate productivity shock, $Z_1 \in \{Z^h, Z^l\}$ with equal probability, and that the productivity reverts to the initial level over time.

We first recover the path of aggregate capital that will arise in the laissez-faire economy after introducing the shock. Given that the borrowing limit is 0 (and there are not short-selling constraints), we do not need to solve a portfolio problem, as households will only invest in capital. We guess and iterate on two paths on capital returns that generate market clearing given the household optimization and aggregation. We assume that the economy is back at the initial steady state levels after H periods (that is, the capital paths have converged back to where they started). The procedure to compute this is similar to what we did in the benchmark exercise. From this step, we recover the laissez-faire sequences of capital, one for each of the two shock paths, $\{K_0^h, K_1^h, ..., K_H^h\}$ and $\{K_0^l, K_1^l, ..., K_H^l\}$. Note that given our shock structure, $K_0^h = K_0^l = K_0$, and $K_1^h = K_1^l$.

[53]For this part, we use a quasi-newton method based on the Jacobian of the aggregate asset function at the initial equilibrium. The computation of the jacobian is discussed in the next subsection.

The government announces a sequence of fiscal policies that implement a sequence of capital and debt $\{K_t^j, B_t\}_{t=0}^H$ for $j = \{h, l\}$ where K remains as in the laissez-faire transition. The paths of B are assumed equal to the path in the benchmark exercise and independent of the shock. As before, we assume that at period H, the economy is in the final stationary equilibrium, with $K_H = K^1$ and $B_H = B^1$.

We compute the transitions as follows. We start with guesses for capital returns and risk-free rates. Given the shock structure, the returns on capital net of depreciation R_t^k and bonds R_t are equal to each other for $t > 1$. This means that we need to guess the paths for the returns to capital $\{R_1^{j,k}, R_2^{j,k}, ..., R_H^{j,k}\}$ for $j = \{h, l\}$ and the risk-free rate from period 0 to 1, R_1, with $R_0^j = R^o$ and $R_H^j = R^1$ for $j = \{h, l\}$. Given these guesses, we can use (16) to obtain the sequence of implied transfers, T_t^j. As above, starting from the value function V^1 and composite x^1 set at the values of the final equilibrium, we use the Euler equation and the value function to iterate backwards up to $t = 1$ and construct sequences of V_t^j and x_t^j for each shock path $j = \{h, l\}$. Note that from $t = 1$ on, each path does not face uncertainty, and therefore the households do not choose a portfolio between capital and bonds.

The problem at $t = 0$, however, contains a portfolio problem, which we solve using a change of variables and by generalizing the endogenous grid method. Let θ_i be household's i share of total savings $a_{i,1}$ allocated to risky capital and $(1 - \theta_i)$ be the share allocated to bonds. At $t = 0$ households choose total savings $a_{i,1}$ and the portfolio θ_i to satisfy an Euler equation and a portfolio equation:

$$x_{i,0}^{-1/\zeta} \geq \beta \left(\mathbb{E}_z V_{i,1}^{1-\gamma}\right)^{\frac{\gamma-1/\zeta}{1-\gamma}} \mathbb{E}_{z,j} \left(V_{i,1}^{1/\zeta-\gamma}((1-\theta_i)R_1 + \theta_i R_1^{k,j})x_{i,1}^{-1/\zeta}\right)$$
$$R_1 \mathbb{E}_{z,j} \left(V_{i,1}^{1/\zeta-\gamma} x_{i,1}^{-1/\zeta}\right) + \bar{\lambda}_i = \mathbb{E}_{z,j} \left(V_{i,1}^{1/\zeta-\gamma}(R_1^{k,j})x_{i,1}^{-1/\zeta}\right) + \underline{\lambda}_i$$

where $\bar{\lambda}_i$ is the multiplier of the constraint that $\theta_i > 0$ and $\underline{\lambda}_i$ is the multiplier of the constraint that $\theta_i < 1$.

In our backward iteration, we arrive at $t = 1$, with $x_{i,1}$ and $V_{i,1}$. We first solve for θ_i using the portfolio equation above by taking into account that $x_{i,1}$ and $V_{i,1}$ depend on total cash-on-hand $\omega_{i,1}$ which is the portfolio return, $\omega_{i,t} = (\theta_i R_1^{k,j} + (1 - \theta_i)R_1)a_1$. Effectively, we perform a change of variables and solve for optimal θ_i to satisfy the portfolio equation, using interpolation. We then iterate back to period 0 and solve for optimal savings, taking into account that optimal θ_i depends on a_1, using the Euler equation.

We now have all the sequences of V_{it}^j and x_{it}^j for all households. With these sequences, we compute the policy functions and the transition function for the distribution of households. We then, starting from Λ^o, iterate forward the evolution of the distribution. With this, we compute the aggregate of the household savings at each time, A_t^j for $j = \{h, l\}$ and also compute the capital demand in period 0, $\mathcal{K}_1$. We then look for a root: sequences $\{R_t^{k,j}\}$ and R_1 such that $A_t^j = B_t^j + K_t^j$ for all $t \leq H$ and $j = \{h, l\}$ and $\mathcal{K}_1 = K_1$ (up to some tolerance).

D.8 Elasticities

The computation of the elasticities we fixed a horizon, H, and set a value $\tau < H$ to be the date where the interest rate changes. We then solve for the sequence of V and η associated with a sequence of interest rates such that $R_t = R^o$ for $t \neq \tau$ and $R_\tau = R^o + \Delta$, by iterating backwards from $t = \tau$ and starting with the laissez-faire equilibrium values. We iterate forward the distribution and compute the implied aggregate savings at each date from $t = 1$ to H, A_t^{up}. We do the same for a sequence of interest rates such that $R_t = R^o$ for $t \neq \tau$ and $R_\tau = R^o - \Delta$, and obtain the sequence of aggregate savings, A_t^{down}. We then compute the (two-sided) numerical derivative, $(A_t^{up} - A_t^{down})/(2\Delta)$ at each time up to H, and use these to construct the elasticities $\xi_{t,\tau}$.[54]

E The Growth Economy

In this appendix, we show how the key expressions of Section 2 are modified by the presence of exogenous labor-augmenting technological growth. The derivations are standard and are included for completeness.

Assume technology is given by

$$Y_t = F(K_t, (1+g)^t L_t),$$

where $g \geq 0$ is the constant rate of growth of labor-augmenting technology. Letting a tilde denote variables divided by $(1+g)^t$, constant returns implies

$$\tilde{Y}_t \equiv (1+g)^{-t} Y_t = F(\tilde{K}_t, L_t).$$

The representative firm's first-order conditions are (dropping t subscripts)

$$\begin{aligned} F_k(\tilde{K}, L) &= \mu(1+\tau^k) r^k \\ F_l(\tilde{K}, L) &= \mu(1+\tau^n)\tilde{w}. \end{aligned}$$

We also have $\widetilde{\Pi} = (1-\tau^\pi)(\mu-1)F(\tilde{K}, L)/\mu$.

Given the absence of a wealth effect on labor supply, we assume that the disutility of working grows at rate g as well (dropping i and t indicators):

$$x(c,n) = c - (1+g)^t v(n),$$

giving us

$$\tilde{x}(\tilde{c}, n) \equiv (1+g)^{-t} x(c,n) = \tilde{c} - v(n).$$

We also assume that the borrowing constraint is scaled by $(1+g)^t$.

We can write the household's problem as

$$\begin{aligned} V_t(a,z,\theta) = \max_{a',n,c} \ & \phi(x(c,n), h(V_{t+1}(a',z',\theta'))) \\ \text{s.t. } & c + a' \leq w_t z n + \theta \Pi_t + (1+r_t)a + T_t \\ & a' \geq (1+g)^{t+1} \underline{a}, \end{aligned}$$

where we have altered the last constraint to account for growth and h is a certainty equivalent

[54] This is what Auclert, Bardóczy, Rognlie, and Straub, 2021 refer to as the "direct method".

operator. The constraint set can be rewritten as

$$\begin{aligned}\tilde{c} + (1+g)\tilde{a}' &\leq \tilde{w}_t z n + \theta\widetilde{\Pi}_t + (1+r_t)\tilde{a} + \tilde{T}_t \\ \tilde{a}' &\geq \underline{a}.\end{aligned}$$

Thus, if (c, n, a') is feasible at time t, then $(\tilde{c}, n, \tilde{a}')$ satisfies the normalized constraint set and vice versa. If we assume ϕ is constant-returns in x and h is homogeneous of degree 1, if $V_t(a, z, \theta)$ satisfies the consumer's Bellman equation, then $\tilde{V}_t(\tilde{a}, z, \theta) \equiv (1+g)^{-t} V_t(a, z, \theta)$ satisfies

$$\tilde{V}_t(\tilde{a}, z, \theta) = \max_{\tilde{c}, n, \tilde{a}'} \phi(\tilde{x}(\tilde{c}, n), (1+g) h(\tilde{V}_{t+1}(\tilde{a}', z', \theta'))),$$

subject to the normalized constraint set, and vice versa.[55]

Note that for an interior optimum for n, the first-order condition can be expressed as follows:

$$v'(n) = z\tilde{w}.$$

Hence, labor supply is constant as long as $\tilde{w}$ remains constant.

The government's budget constraint can be rewritten in normalized form:

$$\tilde{T}_t = \tau_t^n \tilde{w}_t N_t + \tau_t^k r_t^k \tilde{K}_t + \tau_t^\pi \widetilde{\Pi}_t / (1 - \tau_t^\pi) + (1+g)\tilde{B}_{t+1} - (1+r_t)\tilde{B}_t.$$

Let $\tilde{X}_t \equiv \tau_t^n \tilde{w}^o N^o + \tau_t^k r_t^k \tilde{K}_t + \tau_t^\pi \widetilde{\Pi}^o / (1 - \tau_t^\pi)$ denote normalized tax revenue before transfers when keeping after tax normalized wages and profits constant. Following the same steps as the proof of Lemma 1, we have

$$\tilde{X}_t = F(\tilde{K}_t, N^o) - F(\tilde{K}^o, N^o) - (r_t + \delta)\tilde{K}_t + (r^o + \delta)\tilde{K}^o.$$

Condition (iii) of Lemma 1 (equation (3)) becomes

$$(1+g)\tilde{B}_{t+1} - (1+r_t)\tilde{B}_t - \tilde{T}_t \geq F(\tilde{K}^o, N^o) - F(\tilde{K}_t, N^o) - (r^o + \delta)\tilde{K}^o + (r_t + \delta)\tilde{K}_t.$$

Condition (ii) becomes $\tilde{T}_t \geq -(r_t - r^o)\underline{\tilde{a}}$, and condition (i) remains unchanged. Note that in a steady state (that is, relevant aggregates grow at rate g), Condition (iii) becomes

$$(g - r_{ss})\tilde{B}_{ss} - \tilde{T}_{ss} \geq F(\tilde{K}_0, N^o) - F(\tilde{K}_{ss}, N^o) - (r^o + \delta)\tilde{K}^o + (r_t + \delta)\tilde{K}_{ss}.$$

Hence, debt increases government revenues in the steady state as long as $g > r_{ss}$. Expressions in Claims 1 and 2 are adjusted in a similar fashion to obtain normalized equivalents.

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[55]For the simulations, we use $\phi(x, h) = ((1-\beta)x^{1-\zeta} + \beta h^{1-\zeta})^{1/(1-\zeta)}$. In this case, we can define $\tilde{\beta} \equiv \beta(1+g)^{1-\zeta}$ and write $\tilde{\phi}(\tilde{x}, h) = ((1-\tilde{\beta})\chi\tilde{x}^{1-\zeta} + \tilde{\beta}h^{1-\zeta})^{1/(1-\zeta)}$, where $\chi \equiv (1-\beta)/(1-\tilde{\beta})$. This is well defined as long as $\tilde{\beta} \leq 1$. Growth can be accommodated by re-scaling the discount factor, as expected with homogeneous preferences.

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