Online Appendix to "Reputation and Sovereign Default"

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A Existence and uniqueness of a q^o and a continuous q^c

Here we show that given $\{F_\tau\}_{\tau=0}^{\infty}$, there exists a unique q^o and a unique continuous q^c that satisfy the integral equations described in the main body of the paper.

Instead of working with vector-valued operators, the idea of the proof is to substitute the equation for q^o into q^c. Then, to prove the existence, uniqueness and continuity of q^c, we construct a contraction T mapping the space of bounded, continuous functions to itself and where q^c is a fixed point of this mapping.

First, define $T^o\{f\}(\tau)$ as

$$T^o\{f\}(\tau) = \int_0^\infty \Bigg[\left(\int_0^s (i+\lambda)e^{-(i+\lambda)\tilde{s}} \mathrm{d}\tilde{s} + e^{-(i+\lambda)s} f(\tau+s) \right) (1 - F_\tau(\tau+s)) + \int_0^s \left(\int_0^{\tilde{s}} (i+\lambda)e^{-(i+\lambda)\Delta} \mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tau+\tilde{s}) \Bigg] \epsilon e^{-\epsilon s} \mathrm{d}s. \quad (1)$$

In words, $T^o\{f\}(\tau)$ is the value of a bond given an opportunistic government where upon a type switch, the owner receives an arbitrary payoff $f(\cdot) \in [0,1]$.

Next likewise, define $T^c\{g\}(\tau)$ as

$$T^c\{g\}(\tau) = \frac{i+\lambda}{i+\lambda+\delta} + \int_0^\infty e^{-(i+\lambda+\delta)s} g(\tau+s) \delta \mathrm{d}s. \quad (2)$$

In words, $T^c\{g\}(\tau)$ is the value of a bond given a commitment type government where upon a type switch, the owner receives an arbitrary payoff $g(\cdot) \in [0,1]$.

Finally, let $T\{f\}(\tau) \equiv T^c\{T^o\{f\}\}(\tau)$. Here, T is the value of a bond given a commitment type government where upon two type switches (from commitment to opportunistic and back again), the owner receives an arbitrary payoff $f(\cdot) \in [0,1]$.

We now proceed to showing that T^o and T^c are each well defined, and that T is a contraction on the space of bounded continuous functions. First, we can rewrite T^c and T^o as:

$$T^c\{g\}(\tau) = \underline{q} + \delta H_0(-\tau) \int_\tau^\infty H_0(s) g(s) ds$$

$$T^o\{f\}(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} (1 - F_s(s+\hat{s})) e^{-\epsilon\tilde{s}} dH_1(\hat{s}) d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s}) f(s+\tilde{s})(1 - F_s(s+\tilde{s})) d\tilde{s}$$

where

$$\underline{q} = \frac{i+\lambda}{i+\lambda+\delta}, H_0(s) = e^{-(i+\lambda+\delta)s}, H_1(s) = \left(1 - e^{-(i+\lambda)s}\right), H_2(s) = e^{-(i+\lambda+\epsilon)s}$$

and where we used integration by parts to rewrite T^o.

Plugging the equation for T^o back into T^c we obtain that q^c is a fixed point of the operator, T, now written as:

$$T\{f\}(\tau) = g_0(\tau) + \delta\epsilon H_0(-\tau)\int_\tau^\infty \int_0^\infty g_1(s,\tilde{s})d\tilde{s}ds$$
$$\text{where } g_1(s,\tilde{s}) = H_0(s)H_2(\tilde{s})(1 - F_s(s+\tilde{s}))f(s+\tilde{s})$$

and where

$$g_0(\tau) = \underline{q} + \delta\epsilon H_0(-\tau)\int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s)e^{-\epsilon\tilde{s}}(1 - F_s(s+\hat{s}))dH_1(\hat{s})d\tilde{s}ds$$

We now argue that for any bounded non-negative continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s})d\tilde{s}ds$, exists. We show this in three steps.

(a) Given that f is continuous, it follows that the function g_1 is measurable in $\mathbb{R}^2_+$, given our assumption that $F_s(s+\tilde{s})$ is measurable, together with H_0, H_2 and f continuous (g_1 is the product of measurable functions, and thus it is itself measurable).

(b) The integral $\int_0^\infty g_1(s,\tilde{s})d\tilde{s}$ exists given $s \in \mathbb{R}_+$. f non-negative and bounded implies that there exists a $M > 0$ such that $0 \leq f \leq M$. In addition, that $F_s(s+\tilde{s}) \in [0,1]$ implies $0 \leq g_1(s,\tilde{s}) \leq H_0(s)H_2(\tilde{s})M \equiv \bar{g}(s,\tilde{s})$. Given $s \in \mathbb{R}_+$, the function $\bar{g}(s,\cdot)$ is integrable in $\mathbb{R}_+$, and it thus follows that $g_1(s,\cdot)$ is bounded by two integrable functions, and thus it is also integrable.

(c) From the previous step, $0 \leq \int_0^\infty g_1(s,\tilde{s})d\tilde{s}ds \leq \int_0^\infty H_0(s)H_2(\tilde{s})Md\tilde{s}$. That is, the function $g_2(s) = \int_0^\infty g_1(s,\tilde{s})d\tilde{s}$ is bounded between 0 and $\int_0^\infty \bar{g}(s,\tilde{s})d\tilde{s} = \hat{g}(s)$. Given that $\hat{g}(s)$ is integrable in $\mathbb{R}_+$, it provides an integrable upperbound, and it follows that the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s})d\tilde{s}ds$, exists.

A similar argument shows that the iterated integral in the definition of $g_0(\tau)$ exists.

Let B denote the space of continuous functions $f : \mathbb{R}_+ \to [\underline{q}, 1]$ with the sup norm. Note that this is a complete metric space. We make the following two observations about the operator T:

1) T maps B into itself.

   We have already shown that for any bounded non-negative and continous f, $T\{f\}(\tau)$ exists.

Note also that $T\{f\}(\tau) \geq \underline{q} \geq 0$ and

$$T\{f\}(\tau) \leq \underline{q} + \delta\epsilon \underbrace{\left[\int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s-\tau)e^{-\epsilon\tilde{s}} dH_1(\hat{s}) d\tilde{s} ds + \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s}) d\tilde{s} ds\right]}_{=1}$$

where the inequality follows from using that $0 \leq f \leq 1$ and $0 \leq F_s \leq 1$. So $T\{f\} : \mathbb{R}_+ \to [\underline{q}, 1]$.

The continuity of $T\{f\}$ follows from the fact that $g_0(\tau)$ is continuous (as it is the sum a constant and the product of two continuous functions) together with the fact that $\int_\tau^\infty \int_0^\infty g_1(s,\tilde{s}) d\tilde{s} ds$ is an absolutely continuous function of τ.

2) T is a contraction mapping.

Consider two functions f and g. Then we have that

$$T\{f\}(\tau) - T\{g\}(\tau) = \delta\epsilon \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s})(1 - F_s(s+\tilde{s}))(f(s+\tilde{s}) - g(s+\tilde{s})) d\tilde{s} ds$$

Using that $F_s(s+\tilde{s}) \in [0,1]$ we get

$$|T\{f\}(\tau) - T\{g\}(\tau)| \leq |f-g| \epsilon\delta \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s}) d\tilde{s} ds = \frac{\epsilon\delta}{(i+\lambda+\epsilon)(i+\lambda+\delta)} |f-g|$$

Thus T is a contraction mapping with modulus $\frac{\epsilon}{i+\lambda+\epsilon} \times \frac{\delta}{i+\lambda+\delta} < 1$.

It follows by the contraction mapping theorem that there exists a unique bounded and continuous function q^c such that $T\{q^c\} = q^c$ and where $q^c(\tau) \in [\underline{q}, 1]$ for all $\tau \geq 0$.

Given the existence and uniqueness of a continuous function q_c we can substitute back in the q^o equation and obtain the existence and uniqueness of q^o. It is straightforward to show that $q^o(s) \in [0,1]$ for all s.

B Continuity of q^o given construction requirement (16)

We have already shown above that q^c is continuous in any equilibrium. The continuity of q^o cannot be guaranteed in the same fashion (that is, independently of $\{F_\tau\}$). However, we can show that for any family $\{F_\tau\}$ that satisfies our construction requirement in (16), q^o must be continuous.

From the proof in Appendix A, recall that q^o can be written as:

$$q^o(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} (1 - F_s(s+\hat{s}))e^{-\epsilon\tilde{s}} dH_1(\hat{s})d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s})q^c(s+\tilde{s})(1 - F_s(s+\tilde{s}))d\tilde{s}$$

where $H_1(s) = \left(1 - e^{-(i+\lambda)s}\right)$, and $H_2(s) = e^{-(i+\lambda+\epsilon)s}$.

For a family $\{F_\tau\}$ that satisfies our construction requirement in (16), the above implies that $q_o(s) = 0$ for all $s \geq T$, as $F_s(s+\hat{s}) = 1$ for all $s \geq T$ and $\hat{s} \geq 0$.

For all $s \leq T$, we have then that

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} (1 - F_s(\hat{s}))e^{-\epsilon(\tilde{s}-s)} dH_1(\hat{s}-s)d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s}-s)q^c(\tilde{s})(1 - F_s(\tilde{s}))d\tilde{s}$$

which implies that the $\lim_{s\uparrow T} q^o(s) = 0$. Thus q^o is continuous at T.

Finally, using condition (16), and letting $\hat{x}(s) = \frac{x(s)}{1-\rho(s)}$, we have that for $s < T$,

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} e^{-\int_s^{\hat{s}} \hat{x}(\tau)d\tau} e^{-\epsilon(\tilde{s}-s)} dH_1(\hat{s}-s)d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s}-s)q^c(\tilde{s})e^{-\int_s^{\tilde{s}} \hat{x}(\tau)d\tau} d\tilde{s}$$

which guarantees that q^o is a continuous function of s for $s \in [0, T)$.

Hence, we have shown that the function $q^o(s)$ associated with a family of default distributions that satisfy (16) must be continuous for all $s \geq 0$.

C H given by (17) satisfies Assumption 1

We now show that H in equation (17) satisfies the conditions in Assumption 1 given our parameters.

For part(i): Lipschitz continuity. Consider two points $x_0 = (b_0, q_0)$ and $x_1 = (b_1, q_1)$ in $\mathbb{X}$. Let $H_0 = H(b_0, q_0)$ and $H_1 = H(b_1, q_1)$. Let $\tilde{r} = r + \lambda$ and $\tilde{i} = i + \lambda$. Let $[a]^+ = \max\{a, 0\}$, and for our parameters, $\tilde{r} > \tilde{i}$. Then,

$$\begin{aligned}
|H_0 - H_1| &= \left|[\tilde{r} - \tilde{i}/q_0]^+(y - b_0) - [\tilde{r} - \tilde{i}/q_1]^+(y - b_1)\right| \\
&= \left|\left([\tilde{r} - \tilde{i}/q_0]^+ - [\tilde{r} - \tilde{i}/q_1]^+\right)(y - b_0) + [\tilde{r} - \tilde{i}/q_1]^+(b_1 - b_0)\right| \\
&\leq \frac{\tilde{r}^2}{\tilde{i}}|q_0 - q_1| + |\tilde{r} - i| \times |b_0 - b_1| \leq \max\{\tilde{r}^2/\tilde{i}, r^\star - i\} \times (|q_0 - q_1| + |b_0 - b_1|) \\
&\leq \sqrt{2}\max\{\tilde{r}^2/\tilde{i}, r^\star - i\}|x_0 - x_1|
\end{aligned}$$

where the first inequality follows from the facts that (i) $\tilde{r}^2/\tilde{i}$ is the highest (absolute value) slope of the function $g(q) = [\tilde{r} - \tilde{i}/q]^+$ given $\tilde{r} > \tilde{i}$ and (ii) $[\tilde{r} - \tilde{i}/q]^+ \leq \tilde{r} - \tilde{i}$ as $q \leq 1$. The second inequality follows from $a + b \leq \sqrt{2}\sqrt{a^2 + b^2)}$ for $a \geq 0, b \geq 0$. Thus $M \equiv \sqrt{2}\max\{\tilde{r}^2/\tilde{i}, r^\star - i\}$ is the Lipschitz constant for all all $x_0, x_1 \in \mathbb{X}$.

Parts (ii) and (iii): These are immediate.

Parts (iv): In this case, $\underline{q} = \frac{i+\lambda}{r+\lambda}$, as $H(0, q) = 0$ for all $q \leq \underline{q}$ and $H(0, q) > 0$ for all $q > \underline{q}$. Now note that for our parameter values $\underline{q} = 0.6 < \frac{i+\lambda}{i+\lambda+\delta+\epsilon} = 0.875$.

Part (v): $H(\overline{B}, 1) = 0$ given that $\overline{B} = y$.

Part (vi): $H > 0$ requires $q \in (\underline{q}, 1]$ and $b \in [0, y)$. In this case, $H(b, q) = \left(r^{\star} + \lambda - \frac{i+\lambda}{q}\right)(y - b)$ which is differentiable in this domain.