Reputation and Sovereign Default*

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Abstract

This paper presents a continuous-time model of sovereign debt. In it, a relatively impatient sovereign government's hidden type switches back and forth between a commitment type, which cannot default, and an opportunistic type, which can, and where we assume outside lenders have particular beliefs regarding how a commitment type should borrow for any given level of debt and bond price. In any Markov equilibrium, the opportunistic type mimics the commitment type when borrowing, revealing its type only by defaulting on its debt at random times. The equilibrium features a "graduation date": a finite amount of time since the last default, after which time reputation reaches its highest level and is unaffected by not defaulting. Before such date, not defaulting always increases the country's reputation. For countries that have recently defaulted, bond prices and the total amount of debt are increasing functions of the amount of time since the country's last default. For countries that have not recently defaulted (i.e., those that have graduated), bond prices are constant.

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1 Introduction

This paper presents a continuous-time model of sovereign debt where a sovereign government's reputation evolves over time. In the model, a relatively impatient sovereign government's hidden type exogenously switches back and forth between a *commitment* type, which cannot default, and an *opportunistic* type, which can default on the country's debt at any time. We consider the government's reputation at any time to be the international lending markets' Bayesian posterior that the country's government is the commitment (or trustworthy) type.

We show that such a model helps to explain several important characteristics of actual sovereign debt: First, some countries are considered "serial defaulters." In particular, countries that have recently defaulted are considered more likely to default again and pay correspondingly higher interest rates on their debt. In a large historical analysis of sovereign bond returns in the last two centuries, Meyer, Reinhart and Trebesch (2019) confirm that default history matters for sovereign bond prices.[1]

Second, countries that have recently defaulted can sustain much less debt relative to their GDP than countries that have not recently defaulted — a phenomenon referred to as "debt intolerance." Third, some countries do eventually "graduate" into the set of "debt-tolerant" or relatively trusted countries, but only after decades of good behavior. For example, several European economies defaulted frequently during their history before eventually graduating from recurrent external defaults in the 20th century. Reinhart, Rogoff and Savastano (2003) first coined the term "debt intolerance" and discuss the history of serial defaulters and graduation.[2]

Finally, default events and variations in interest rate spreads are weakly associated with fundamentals such as debt and output ratios; they cannot be precisely predicted.[3] Our model captures each of these characteristics.

We present a Markov equilibrium for our model where a borrower country's debt and the

[1] Several other papers have documented a positive relationship between default history and subsequent bond yields, although for smaller samples. The recent empirical literature emphasizes that the effects of default (in particular those that feature large "haircuts", that is, large investors' losses) on future yields are quantitatively large and persistent. This fact was first highlighted by Cruces and Trebesch (2013), see also Asonuma (2016).

[2] Their main point is that indeed "a country's record of meeting its debt obligations [...] in the past is relevant to forecasting its ability to sustain moderate to high levels of indebtedness." They show that "safe" thresholds of debt for serial defaulters are low, especially when compared to advanced economies and emerging markets with no history of default. In addition, they show that graduation is rare, and highlighted, in 2003, Greece, Portugal, Malaysia, Thailand and Chile as potentially recent graduates, although Greece eventually did default. In more recent work, Qian, Reinhart and Rogoff (2011) emphasize that "graduation from recurring sovereign external debt crises is a very tortuous process that sometimes takes a century or more". They show that 20 years without a default is an important marker, as two-thirds of recurrent sovereign default have occurred within that window. But 50 years or more is necessary to "meaningfully speak of graduation".

[3] See Tomz and Wright (2007) and Aguiar and Amador (2014). This fact has led some researchers to postulate the need for self-fulfilling debt crisis as a necessary ingredient in models of sovereign debt, as in Aguiar, Chatterjee, Cole and Stangebye (2016).

price of its bonds are both strictly increasing functions of the amount of time since its last default (and a country can issue new bonds *immediately* after a default, albeit at high interest rates). In fact, the highest-priced bonds are those issued by the countries with the *most* debt (the countries that have experienced the longest amount of time since a default).[4] This occurs because, from the perspective of a lender, the probability of default is a strictly decreasing function of the amount of time since the last default. Thus, our equilibrium displays both debt intolerance (high interest rates paid by countries with low debt levels) and serial default (a relatively high probability of default by countries that have recently defaulted). Finally, our equilibrium features a "graduation date" — an amount of time since the last default T such that if a government goes T amount of time without defaulting, foreign lenders become certain it is the commitment type (and offer the lowest interest rate on its debt). At this point, the reputation of the country and the price of its bonds never change as long as default does not occur. This conceptualization of a "graduation date", or the point at which a country joins the set of debt-tolerant or trusted countries, we believe is novel to this paper.

Games where informed players (in our case, a government who knows its type) have rich action spaces are notoriously difficult to characterize. We make progress by initially making the default decision of the opportunistic type the only strategic choice. That is, we assume foreign lenders have *particular* beliefs regarding how a commitment type *should* borrow for any given level of debt and bond price (and believe that the government must be the opportunistic type if it deviates from this borrowing behavior). Given this assumption, the *commitment type* plays a passive role, following its expected behavior. (In a later section we relax this assumption and allow debt issuance to be a strategic choice of both types.)

The *opportunistic type*, on the other hand, faces at all times the strategic decision of whether to mimic the commitment type or *default*, which we initially model as wiping out all of its existing debt. The benefit of defaulting is, of course, that the country makes no further coupon payments on its debt. The cost is that outside lenders, at the time of default, become certain the country is the opportunistic type. (Unlike much of the quantitative sovereign debt literature, we assume no *direct* costs of default.)[5]

We first show for any *particular* specification of outside lender expectations regarding how a commitment type should act, how to solve for a Markov equilibrium as the solution to an ordinary differential equation. We next show that as long as these expectations are in a reasonable class (namely, the country increases its debt by less if it already has high debt, borrows more if it faces better bond prices, and borrows a strictly positive amount if it has no debt and a good

[4] A relevant example of this would be Japan.

[5] The presence of the commitment type allows us to avoid the Bulow and Rogoff (1989) result of no debt in equilibrium absent direct default costs. For an alternative analysis of reputation, based on trigger strategies, see Kletzer and Wright (2000) and Wright (2002).

enough bond price), *all such equilibria look qualitatively identical* — the exact specification of lender expectations does not qualitatively matter.

We show that in our Markov equilibrium, for an endogenous finite period of time T after a default, an opportunistic government sets a positive but finite hazard rate of defaulting which depends only on how much time has occurred since the last default. Further, as this amount of time since the last default approaches T, this arrival rate of default, *conditional on the government being the opportunistic type,* approaches infinity (certain immediate default).

Nevertheless, from the perspective of a foreign lender, the probability of default is *decreasing* in the amount of time since the last default. This happens because although an opportunistic government is *more* likely to default the longer it has been since the last default, whether the country's government actually *is* the opportunistic type is *less* likely the longer it has been since a default, and this latter effect dominates. This implies that if the amount of time since the last default is positive but less than T, the country has an interior reputation that increases over time but after T is certainly the commitment type.

A country's reputation increases in the amount of time since its last default because the commitment type defaults with zero probability and the opportunistic type defaults not only with positive probability but with a probability high enough to counter any drift in reputation due to exogenous type switching.

But this mixing imposes strict requirements on the path of play. In particular, we show this willingness to mix implies that the opportunistic type must receive constant *net* payments from foreign lenders. We show that the opportunistic types never, on net, repay (instead making coupon payments by issuing new debt). But that foreign lenders always break even in expectation then implies that future commitment types must be the ones paying back the debt. That is, in equilibrium, opportunistic types extract constant rents from future commitment types. Only a commitment type will run a trade surplus or a trade deficit smaller than the one extracted in equilibrium by the opportunistic type. In fact, opportunistic types default with probability one once they cannot extract this constant rent. They do not wait for the trade deficit to actually become negative before defaulting.[6]

These characterizations are derived assuming the borrower country starts the game with zero debt and zero reputation (outside lenders are convinced it is the opportunistic type). This is the relevant subgame after the first default. In this subgame there is, in equilibrium, a specific debt level associated with each reputation, which we call the country's *appropriate* debt level. We next consider starting the game with arbitrary values of reputation and debt. We show that if

[6]In a recent paper, Aguiar, Chatterjee, Cole and Stangebye (2017) exploit equilibrium mixing where the government is indifferent between default or not, to generate dynamics of debt and spreads that more closely match the data, improving the fit of benchmark quantitative Eaton and Gersovitz (1981) sovereign debt model.

the country's debt is above the level appropriate for its initial reputation, the equilibrium calls for immediate probabilistic default. If the country defaults, the game reverts to the subgame with zero debt and zero reputation. If the country does not default, its debt stays the same, but the country's reputation (from the fact that it did not default when it was supposed to with positive probability) jumps to its appropriate level. Next, we show that if the country's debt starts below its appropriate level, the equilibrium calls for the opportunistic government to set the probability of default to zero for a finite amount of time until its reputation and debt converge to the appropriate levels.

The model in this paper shares several features with the taxation paper of Phelan (2006), (which itself builds on the reputation papers of Barro (1986), Kreps and Wilson (1982), and Milgrom and Roberts (1982)). In that paper, a government can be either a commitment type, which must tax output at a low rate, or an opportunistic type, which taxes either the low rate or confiscates all output, with exogenous hidden type switches as in the model presented here. Like this paper, the opportunistic type mixes for some time and then separates from the commitment type in the Markov equilibrium. Apart from these two characteristics, however, the implications of the two models differ. In this paper, we explicitly model the relation between *debt*, a payoff-relevant state variable, and reputation, and our characterization relies heavily on the specific features of debt contracts. Contrary to taxing, selling debt has an explicit time dimension: it is always an ex-post bad thing. This time dimension then drives our predictions for the paths of interest rates, debt levels, and default properties not present in earlier work. One striking difference is that in this paper, except for before the first default or off the equilibrium path, *there is no value to a good reputation* — the government's reputation affects government borrowing and the price of its debt, but not its continuation value.[7]

The models of Cole, Dow and English (1995), Alfaro and Kanczuk (2005), and D'Erasmo (2011) also feature alternating government types in a sovereign debt context. In Cole et al. (1995) and Alfaro and Kanczuk (2005), "good" governments try to differentiate themselves from "bad" governments (those that always default).[8] In Cole et al. (1995), a "good" government signals his type by settling previously defaulted debt. In Alfaro and Kanczuk (2005), a "good" government signals his type by not defaulting. In our environment, the focus is instead on the opportunistic government behavior decision not to default. In addition, in both of these papers, the authors focus on the dynamics of beliefs (while the debt level remains constant). A contribution of our paper is

[7] Other recent models on reputation building include the discrete time models of Liu (2011) and Liu and Skrzypacz (2014), and the continuous time models of Faingold and Sannikov (2011), Board and Meyer-ter Vehn (2013), and Marinovic, Skrzypacz and Varas (2018).

[8] There is a large literature on quantitative sovereign debt models based on the incomplete markets framework of Eaton and Gersovitz (1981). This literature started with Aguiar and Gopinath (2006) and Arellano (2008), and (mostly) focuses on the business cycle patterns of spreads and default. See Aguiar and Amador (2014) for a review.

to show that the level of debt and a country's reputation are linked in equilibrium. Our paper also differs from D'Erasmo (2011) in that we impose no exogenous cost of default and are able to provide a complete characterization of the equilibrium. In complementary work, Egorov and Fabinger (2016) share our objectives of studying reputation, graduation, debt and interest rate dynamics, and do so in a model where an unchanging government has private information about the realizations of a default cost process. Dovis (2019) studies a model where the government has private information about the realizations of a productivity parameter, and shows that such a model can rationalize periods of exclusion after default.[9]

Section 2 presents our model, where we initially focus on the particular starting conditions of both debt and reputation having zero values. In Sections 3 and 4, we define, construct and prove the existence of a Markov equilibrium of this game. In Section 5, we present a computed example. In Section 6, we show that the characteristics of the equilibrium derived in Section 4 hold for *all* Markov equilibria where the commitment type follows an expectation rule. In Section 7, we relax the assumption that the game starts with zero debt and zero reputation and characterize play of the game for all initial starting conditions for debt and reputation. In Section 8, we allow debt issuances to be a strategic choice for both types. We first show subject to a reasonable condition, that Markov equilibria where governments reveal their types through debt issuances don't exist in our framework. We then use our results from Section 7 to show that the opportunistic type will choose to reveal its type only by defaulting (and otherwise mimic the commitment type) and that the commitment type will follow the expectational rule if it is sufficiently impatient. That is, we show our *pooling* equilibrium is indeed an equilibrium (with sufficient impatience for the commitment type) in that we check the incentives over all deviations as opposed to just those associated with default. Section 9 concludes. An appendix collects the proofs.

2 The Environment

Time is continuous and infinite. There is a small open economy whose government is endowed with a constant flow y of a consumption good. There is a countable list of potential *governments* of the small open economy, with alternating types. With probability ρ_0, the first government on the list is the *commitment type* and with probability $(1-\rho_0)$, the first government on the list is the *opportunistic type*. The list then alternates between types. At any date $t \geq 0$, only one of the potential governments is in charge. With Poisson arrival rate ϵ, an opportunistic type government is replaced by the next government on the list (a commitment type). With

[9] See also Paluszynski (2017) for an analysis of how introducing learning about fundamental shocks improves the performance of the standard sovereign debt model during the European debt crisis. Other papers that deal with the effects of private information in investment, debt sustainability, and maturity structure are Sandleris (2008), Phan (2017), and Perez (2017).

arrival rate δ, a commitment type government is replaced by an opportunistic type. Such switches are private to the government. (We see such switches as private changes in the preferences of leaders, or unobservable changes in the relative influence of different constituencies). We assume international financial markets are populated by a continuum of risk-neutral investors, who discount the future at rate $i > 0$.

The economy has, at any time t, an amount $b(t)$ of outstanding debt held by these risk-neutral investors. The bonds are of long duration and have a coupon that decays exponentially at rate λ, an exogenous parameter controlling the maturity of the bonds. Without loss of generality, we set the coupon t units of time from issuance to $(i+\lambda)e^{-\lambda t}$ to ensure that in equilibrium, the price of a bond is one if default cannot occur.[10]

Initially, we assume that at time $t = 0$ there is no debt, $b_0 = 0$, and the government is known to be the opportunistic type, that is, $\rho_0 = 0$. We will later consider the case where initial debt and initial reputation are not each set to zero. As time progresses, the government's debt, its reputation, and the price of its bonds will evolve as well.

2.1 Strategies

By assumption, a commitment type never defaults on its debts and will make any coupon payments that are due as long as it is in power. An opportunistic type, however, can default. Once a default occurs, we assume that the current bond holders get no additional payments, and the stock of outstanding debt is set to zero. By defaulting, however, an opportunistic government reveals its type and thus sets its reputation ρ (its probability of being the commitment type) to zero.[11]

We further assume that the strategies are *Markov*: that continuation strategies are only a function of the level of debt $b(t)$ and the reputation $\rho(t)$. But then note that since both $b(t)$ and $\rho(t)$ are reset to zero (their initial values) upon default, making continuation strategies contingent only on $b(t)$ and $\rho(t)$ is the same as making continuation strategies contingent only on the amount of time since the last default (which we label τ) since $b(t)$ and $\rho(t)$ would themselves depend only on the amount of time since the last default. Essentially, the assumptions that $b_0 = \rho_0 = 0$ and Markov strategies, along with only opportunistic types being able to default,

[10] This timing is the limit, as the distance between *clock ticks* goes to zero, of a discrete time game where clock ticks occur at times $t \in \{0, \overline{\Delta}, 2\overline{\Delta}, \ldots\}$ and type switches occur between clock ticks. Here, we assume a bond is a promise to pay a stream $\{(i+\lambda)\overline{\Delta}, (i+\lambda)\overline{\Delta}(1-\lambda\overline{\Delta}), (i+\lambda)\overline{\Delta}(1-\lambda\overline{\Delta})^2, \ldots\}$ starting in $\overline{\Delta}$ units of time. Just before each clock tick, if the government is the opportunistic type, it switches to the commitment type with probability $\epsilon\overline{\Delta}$, and if it is the commitment type, it switches to the opportunistic type with probability $\delta\overline{\Delta}$, where $\overline{\Delta} \leq \min\{1/\epsilon, 1/\delta, 1/\lambda\}$ to ensure valid switching probabilities and positive coupon payments.

[11] This is the limit as $\overline{\Delta} \to 0$ of a discrete time timing where after a clock tick, first debt is issued and coupon payments are made, then a default occurs or not, and then a type switch occurs or not. Thus if a default occurs, the government has reputation $\epsilon\overline{\Delta}$ at the beginning of the next period.

ensure that default *restarts the game*. Thus, as long as we are assuming $b_0 = \rho_0 = 0$, we make strategies not a function of history or of debt or reputation, but simply a function of the time since the last default, τ. (In Section 7, where we relax the assumption that $b_0 = \rho_0 = 0$, the state variables become (b, ρ) explicitly again.)

For the commitment type, we assume that as long as it is in control, it follows a pre-specified expenditure rule determined by the expectations of international financial markets of how a commitment type should act. That is, as long as the commitment type is in control, the stock of debt evolves according to

$$b'(\tau) = H(b(\tau), q(\tau)) \tag{1}$$

for some exogenous function H, where $q(\tau)$ represents the price of a bond τ periods after the last default.[12] We will later find conditions on the payoffs for the commitment type and expectations of outside lenders such that the commitment type finds it optimal to follow H.

It follows from the sequential budget constraint that $c(\tau) = y-(i+\lambda)b(\tau)+q(\tau)(b'(\tau)+\lambda b(\tau))$, and thus consumption for the commitment type is determined by $c(\tau) = C(b(\tau), q(\tau))$ where the function C is given by

$$C(b, q) \equiv y - (i + \lambda)b + q(H(b, q) + \lambda b)$$

We restrict attention to debt levels that remain bounded by a finite positive constant $\overline{B}$. We require that $\overline{B} < y/(i+\lambda)$, to guarantee that positive consumption is always feasible at any price.[13] We further impose the following further conditions on $H(b, q)$ (and thus, implicitly, $C(b, q)$):

Assumption 1. *Let $\mathbb{X} \equiv [0, \overline{B}] \times [0, 1]$. The function $H : \mathbb{X} \to \mathbb{R}$ satisfies the following: (i) H is Lipschitz continuous; (ii) H is weakly decreasing in b; (iii) H is weakly increasing in q; (iv) There exists $\underline{q} \in (0, \frac{i+\lambda}{i+\lambda+\delta+\epsilon})$ such that $H(0, q) = 0$ for all $q \in [0, \underline{q}]$, and $H(0, q) > 0$ for all $q \in (\underline{q}, 1]$; (v) $H(\overline{B}, 1) \leq 0$; (vi) H is differentiable in the set of $(b, q) \in \mathbb{X}$ such that $H(b, q) > 0$.*

Restriction (i) is used to guarantee the existence and uniqueness of solution to the differential equation (1). Restrictions (ii) and (iii) guarantee that the commitment type increases its debt by more the higher the price and the lower the inherited debt stock. Restriction (iv) is an impatience restriction. It guarantees that the commitment type is expected to borrow when it has no debt

[12] Fiscal rules that are functions of the inherited stock of debt are widely used in the debt sustainability literature, see Lorenzoni and Werning (2019). Such rules are usually stated without allowing for a feedback from the interest rate. For countries that suffer from sovereign debt crisis episodes and volatile interest rate spreads , it seems natural to allow that the fiscal reaction rule be also affected by the interest rate. For example, in recent work, Gourinchas, Philippon and Vayanos (2017) while studying the Greek debt crisis assume the government follows a fiscal rule where government spending reacts negatively to both the debt level and the interest rate. See also Martin and Philippon (2017) for a multi-country analysis of the European debt crisis with a similar fiscal rule.

[13] In particular, this rules out the possibility that the price is so low that the commitment type cannot feasibly repay its debts. With this assumption, it is always feasible to pay the coupons of the debt, even at a zero price.

and the interest rate equals $i + \delta + \epsilon$ (where i is the world interest rate and δ and ϵ are the arrival rates of the type switches). Restriction (v) guarantees that for a fixed price, $q \in [0, 1]$, $H(\overline{B}, q) \leq 0$ and the differential (1) has a solution that stays within the set $[0, \overline{B}]$ given a fixed price. This is sufficient to guarantee also that, for a fixed price, starting from any initial debt value, the solution converges to a constant debt level, and convergence is monotone. The final restriction (vi) is used to show the existence of an equilibrium.

The above assumption does not impose that H delivers a non-negative consumption $C(b, q)$ for debt and price pairs in $\mathbb{X}$. Such an explicit restriction could be added without further impact in the arguments that follow.[14] Assumption 1 also does not impose a bound on how negative H can be, potentially allowing for bond buybacks. Later on in Section 8, we restrict attention to the case where the government cannot buy back its debts, and thus new bond issuances are non-negative, requiring the additional restriction that $H(b, q) + \lambda b \geq 0$ for all $(b, q) \in \mathbb{X}$.[15] However, such restriction is not needed for all of the results before Section 8.

For the *opportunistic* type, in addition to the Markov restriction, we impose for now a restriction that it always chooses a level of borrowing (and thus consumption) that is identical to that which would have been chosen by a commitment government facing the same debt and price. With this restriction, the only decision left under the control of the opportunistic government is whether to default or not. We will later show that this restriction is without loss of generality: an opportunistic government will have no incentive to reveal itself by choosing a level of borrowing or consumption different from the commitment government, without simultaneously defaulting on its debt.

Given this restriction, we assume that a strategy for an opportunistic government that has just taken power in period τ is a right-continuous and non-decreasing function $F_\tau : \mathbb{R}_+ \longrightarrow [0, 1]$, where $1 - F_\tau(s)$ defines the probability that this government does not default between period τ and $s \geq \tau$ inclusive, conditional on it remaining in power from τ to s. We assume the function $F_\tau(\tau + s)$ is measurable with respect to $(\tau, s) \in \mathbb{R}_+^2$ and let Γ denote the set of all such functions.

This formulation allows both jumps and smooth decreases in the survival probability, $1 - F_\tau$. If F_τ jumps up at $s \geq \tau$, this implies a strictly positive probability of defaulting at exactly date s. When F_τ smoothly increases, the probability of defaulting at exactly date s is zero. In this case, $F'_\tau(s)/(1 - F_\tau(s))$ represents the hazard rate of default at that date (where $F'_\tau(s)$ represents the right derivative at s).

Our Markov restriction imposes that an opportunistic government that takes power in period τ follows a strategy that any previous opportunistic government would also follow from period

[14]Basically, it requires to impose the additional restriction that $H \geq \frac{(i+\lambda)b-y}{q} - \lambda b$. The right hand side of this inequality is always non-negative for any $(b, q) \in \mathbb{X}$ given $b \leq \overline{B} < y/(i + \lambda)$,

[15]Note that this allows for the stock of debt to decrease over time, at rate bounded by the decay parameter λ.

τ onward if it were to remain in power up to period τ without defaulting (since both cases have the same debt and reputation). That is,

Definition 1. *A* **Markov strategy profile** *for opportunistic governments is a collection* $\{F_\tau\}_{\tau=0}^{\infty}$ *with* $F_\tau \in \Gamma$ *for all* τ*, such that*

$$1 - F_\tau(s) = (1 - F_\tau(m^-))(1 - F_m(s)) \text{ for all } 0 < \tau \leq m \leq s, \tag{2}$$

where $F_\tau(m^-) = \lim_{n \to m^-} F_\tau(n)$.

The above restriction implies that the function F_s is pinned down by F_0 for all s such that $F_0(s) < 1$. That is, if there is a strictly positive probability that the opportunistic government at time 0 reaches time s without defaulting, then it is possible to use the conditional probabilities inherent in F_0 to determine F_s. This however fails for s such that $F_0(s) = 1$, as in that case the opportunistic government at time 0 will not reach date s without defaulting first, and thus conditional probabilities are not defined. However, it is still necessary to define how an opportunistic government behaves at such dates, as there is a positive probability that an opportunistic government is in power at such a date due to two or more government type switches.

2.2 Payoffs

If the government does *not* default at period τ, it issues additional bonds $H(b(\tau), q(\tau)) + \lambda b(\tau)$ at endogenous price $q(\tau)$ and its consumption is $C(b(\tau), q(\tau))$. If the government *defaults*, then the game starts over (τ is reset to zero) with $b_0 = 0$. There are no direct costs of choosing to default and no restrictions on government borrowing from then on. In particular, it issues additional bonds $H(0, q(0))$ at endogenous price $q(0)$ and its consumption is $C(0, q(0))$.

The opportunistic type receives a flow payoff equal to $u(c(\tau))$ as long as it is continuously in power, and discounts future payoffs at rate $r > 0$. We assume that $u : \mathbb{R}_+ \to [\underline{u}, \overline{u}]$ for some finite values $\underline{u}$ and $\overline{u}$, and that u is strictly increasing. We make no other assumptions on the preferences of the opportunistic type. (A preview of our results is that our constructed Markov equilibrium is essentially *independent* of u and r. Other than more is preferred to less, and now is preferred to later; the preferences of the opportunistic type will not matter at all.)

2.3 Beliefs

As noted, $\rho(\tau)$ represents the international market's beliefs that the government at period τ is the commitment type *after* it has not defaulted τ periods since the last default. We assume $\rho(\tau)$ is determined by Bayes' rule. (If the government defaults at period τ, Bayesian updating implies ρ and thus τ immediately jump to zero.)

By Bayes' rule, the probability that the commitment type is in power τ periods after the last default is

$$\rho(\tau) = \frac{\text{Probability of no default in } [0,\tau] \text{ and commitment type in power at } \tau}{\text{Probability of no default in } [0,\tau]}$$

As long as the probability of observing no default in $[0,\tau]$ is strictly positive, Bayes' rule pins down the market belief. Because the government type may switch in any positive interval, the only case where the denominator in the above equation is zero is when the opportunistic government defaults immediately at $\tau = 0$, that is when $F_0(0) = 1$ (a case that we handle below). It is helpful to write the evolution of ρ recursively in the following manner.

Consider first the case where $F_0(0) < 1$. Recall $F_\tau(\tau)$ is the probability that the opportunistic government, conditional on being in power τ periods since the last default, defaults at exactly period τ. Let $\rho(\tau^-) \equiv \lim_{n\to\tau^-} \rho(n)$ for $\tau > 0$ and $\rho(0^-) \equiv 0$. This represents the market belief an instant before the current default outcome.

If $F_\tau(\tau) > 0$, ρ jumps from $\rho(\tau^-)$ to

$$\rho(\tau) = \frac{\rho(\tau^-)}{\rho(\tau^-) + (1-\rho(\tau^-))(1-F_\tau(\tau))}. \tag{3}$$

Note this implies if $F_\tau(\tau) = 1$ and $\rho(\tau^-) > 0$ that ρ jumps from $\rho(\tau^-)$ to 1 at τ. Further note this implies if $F_\tau(\tau) = 0$, then ρ does not jump at τ since $\rho(\tau) = \rho(\tau^-)$.

If $F_\tau(\tau) = 0$, reputation ρ still moves, albeit continuously. First, even if the hazard rate of default is zero (that is, $F'_\tau(\tau) = 0$), probabilistic type switches cause ρ to drift towards $\epsilon/(\epsilon+\delta)$ (its unconditional long-run mean). Second, if the hazard rate of default is positive (that is, $F'_\tau(\tau) > 0$), not experiencing default increases the drift in ρ, as not defaulting is informative about the government type. Bayesian updating in this case implies that at points of differentiability[16]

$$\rho'(\tau) = (1-\rho(\tau))\epsilon + \rho(\tau)((1-\rho(\tau))F'_\tau(\tau) - \delta). \tag{4}$$

[16] For small time interval $\Delta > 0$ and a constant hazard rate of default $F'_\tau(\tau)$,

$$\rho(\tau+\Delta) \approx (1-\delta\Delta)\frac{\rho(\tau)}{\rho(\tau)+(1-\rho(\tau))(1-F'_\tau(\tau)\Delta)} + \epsilon\Delta\left(1 - \frac{\rho(\tau)}{\rho(\tau)+(1-\rho(\tau))(1-F'_\tau(\tau)\Delta)}\right).$$

To see this, assume that the government's type stays constant on the interval $[\tau, \tau+\Delta)$ and switches at $\tau+\Delta$ from the commitment type to the opportunistic type with conditional probability $\delta\Delta$, and from the opportunistic to the commitment type with conditional probability $\epsilon\Delta$. The term $\rho(\tau)/(\rho(\tau)+(1-\rho(\tau))(1-F'_\tau(\tau)\Delta))$ is then the belief, conditional on no default between τ and $\tau+\Delta$, that the government is the commitment type just before $\tau+\Delta$. Thus the first term is the probability the government was the commitment type just before $\tau+\Delta$ and didn't switch at $\tau+\Delta$, and the second term is the probability the government was the opportunistic type just before $\tau+\Delta$ and did switch at $\tau+\Delta$; the two ways the government can be the commitment type at $\tau+\Delta$. The derivative of this expression with respect to Δ evaluated at $\Delta = 0$ is equation (4).

Finally, consider the case where $F_0(0) = 1$. As noted above, Bayes' rule does not apply, as the probability of not observing default at exactly date 0 is zero. In this case, we let $\rho(0)$ (the belief *after* no default is observed at date zero) be a free variable. Given this belief $\rho(0)$, equations (3) and (4) continue to hold and determine the evolution of beliefs at all subsequent dates.

2.4 Prices

Let $q(\tau)$ denote the price of the bond if there has not been a default for τ periods (not including no default at exactly date τ). Given the risk neutrality assumption on the foreigners, the price solves

$$q(\tau) = \rho(\tau^-)q^c(\tau) + (1 - \rho(\tau^-))q^o(\tau), \tag{5}$$

where $q^c(\tau)$ and $q^o(\tau)$ denote the price if there has not been a default for τ periods (not including exactly date τ) and the commitment type and opportunistic type are known to be in power, respectively. These prices must lie in $[0, 1]$ and solve the following recursion, given a default strategy F_τ for the opportunistic type: First,

$$q^c(\tau) = \int_0^\infty \left(\int_0^s (i + \lambda)e^{-(i+\lambda)\tilde{s}}\mathrm{d}\tilde{s} + e^{-(i+\lambda)s}q^o(\tau + s) \right) \delta e^{-\delta s}\mathrm{d}s. \tag{6}$$

Here, the outer integral is the expectation over the first type switch. The variable s in the outer integral represents the date of the first type switch from commitment to opportunistic. The two terms in the parentheses calculate the value of the bond conditional on s. The first term is the date τ value of the coupon stream between τ and $\tau + s$. The second term is the date τ value of the remaining bond at date $\tau + s$ conditional on a type switch to an opportunistic government at that time.

Second,

$$q^o(\tau) = \int_0^\infty \Bigg[\left(\int_0^s (i + \lambda)e^{-(i+\lambda)\tilde{s}}\mathrm{d}\tilde{s} + e^{-(i+\lambda)s}q^c(\tau + s) \right) (1 - F_\tau(\tau + s)) + \int_0^s \left(\int_0^{\tilde{s}} (i + \lambda)e^{-(i+\lambda)\Delta}\mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tau + \tilde{s}) \Bigg] \epsilon e^{-\epsilon s}\mathrm{d}s. \tag{7}$$

Here, the outer integral is again the expectation over the first type switch. The terms in the square brackets calculate, again, the value of the bond conditional on s. With probability $(1 - F_\tau(\tau + s))$, default does not occur before date s, and the terms in parentheses are similar to equation (6), but this time using q^c instead of q^o. The last term handles the case where default occurs before the type switch. The outer integral of this term is the expectation over the default date $\tilde{s}$, and the inner integral calculates the value of the coupon payments up to that date.

Note that its integral form implies that $q^c(\tau)$ is continuous (a result shown in Appendix C). Whether $q^o(\tau)$ is continuous or not depends on the strategy profile $\{F_\tau\}$.

When $F_\tau(\tau) = 0$, the above implies that $q(\tau)$ obeys the following differential equation, at points of differentiability:

$$\underbrace{\left[i + \lambda + (1 - \rho(\tau))F'_\tau(\tau)\right]}_{\text{effective discount rate}} q(\tau) = \underbrace{(i + \lambda)}_{\text{coupon}} + \underbrace{q'(\tau)}_{\text{capital gain}} . \tag{8}$$

3 Markov Equilibria

We will focus attention on Markov equilibria where both government types follow the debt accumulation rule $H(b, q)$ (and later verify conditions such that each type wishes to do so). The definition of a Markov equilibrium is:

Definition 2. *A **Markov equilibrium** is a strategy profile for opportunistic governments $\{F_\tau\}_{\tau=0}^\infty$, together with debt, its price and reputation, (b, q, ρ), as functions of time since last default, such that*

1. *(Foreign investors break even in equilibrium.) q and ρ satisfy (5) for some $q^c : \mathbb{R}^+ \longrightarrow [0, 1]$ and $q^o : \mathbb{R}^+ \longrightarrow [0, 1]$ that solve equations (6) and (7).*

2. *(Market beliefs are rational.) $\rho : \mathbb{R}^+ \longrightarrow [0, 1]$; equation (4) holds for all $\tau \geq 0$. For $\tau > 0$ and $\tau = 0$ if $F_0(0) < 1$, equation (3) holds if $F_\tau(\tau) > 0$.*

3. *(Debt evolution and budget constraint.) The level of debt, conditional on no default, evolves according to the pre-specified expenditure rule H:*

$$b(\tau) = \int_0^\tau H(b(s), q(s))ds, \tag{9}$$

for all $\tau \geq 0$ and remains non-negative and bounded, $b(\tau) \in [0, \overline{B}]$.

4. *(opportunistic type optimizes.) For all times since the last default $\tau \geq 0$, the opportunistic government's continuation strategy F_τ maximizes its forward looking payoff taking the path of b and q as given. That is, $\{F_\tau\}_{\tau=0}^\infty$ solves the following collection of optimal control problems (indexed by τ):*

$$V(\tau) = \sup_{F_\tau \in \Gamma} \int_0^\infty \left(\int_\tau^t e^{-(r+\epsilon)(s-\tau)} u(C(b(s), q(s)))ds + e^{-(r+\epsilon)(t-\tau)} V(0) \right) dF_\tau(t).$$

5. *(Markov refinement.)* $\{F_\tau\}_{\tau=0}^{\infty}$ *satisfies the conditions in Definition 1.*

To clarify, Condition 4 does not allow for any "commitment" to future strategies by the opportunistic type, as each opportunistic government takes as given the future paths of prices and debt. That is, altering future default probabilities does not feed into the prices that the date τ government faces. The connection between default probabilities and prices is imposed through different equilibrium conditions (in this case, Conditions 1 and 2).

4 A Markov Equilibrium: Construction and Existence

In this section, we construct a Markov equilibrium as a solution to an ordinary differential equation. In the next section, we show that any Markov equilibrium must also solve this construction.

The main idea for the equilibrium construction is to conjecture that there exists a finite time since the last default, T, such that before T, an opportunistic government sets a strictly positive but finite hazard rate of default (that is, $F_\tau(\tau) = 0$ and $F'_\tau(\tau) > 0$ for $\tau < T$), and consumes at a constant level $c^\star > y$. After and including time T, it defaults immediately, and thus the reputation of surviving after T is at its maximum, $\rho = 1$. (Our continuous time setup implies that if an opportunistic type defaults, it consumes $c(0) = c^\star$ immediately).

This formulation (constant consumption while default is less than certain) guarantees a constant continuation value for the opportunistic type, which keeps this government type indifferent between defaulting or not. We later show such indifference is necessary for equilibrium.

First, let $Q(b, c)$ denote the price which causes a commitment type with debt b to set its consumption to c. That is, $Q(b, c)$ is such that $C(b, Q(b, c)) = c$ for $c \in (y, \bar{c})$ where $\bar{c} \equiv C(0, 1)$. Assumption 1 guarantees that there is a unique such function Q defined on $[0, \bar{b}(c)] \times (y, \bar{c})$ (where $\bar{b}(c)$ is such that $C(\bar{b}(c), 1) = c$). Assumption 1 also guarantees that $Q(b, c)$ is strictly increasing in b and c, reflecting the fact that to maintain a level of consumption higher than y, the bond price must be higher at a higher debt level (as the government must be generating positive revenue from new issuances to sustain $c > y$); and that a higher consumption requires a higher bond price, given a debt level. Note this implies that $Q(b, c) \in (\underline{q}, 1]$ for $c \in (y, \bar{c})$ and $b \in [0, \overline{b}(c)]$. Assumption 1 also implies that Q is differentiable in both arguments for $c > y$ (a result shown in Lemma 4 in the Appendix).

We let $\overline{q} \equiv \frac{i+\lambda}{i+\lambda+\delta}$, which is the bond price consistent with a constant arrival rate of default equal to δ. As we will see, $\overline{q}$ is the highest possible equilibrium price, and that from Assumption 1, $\overline{q} > \underline{q}$.

Construction of $(b(\tau), q(\tau), \rho(\tau))$ for $\tau < T$. For consumption to be constant, paths of bond prices $q(\tau)$ and debt levels, $b(\tau)$, must solve $C(b(\tau), q(\tau)) = c^\star$ for $c^\star \in [0, \bar{c}]$, and thus $q(\tau) =$

$Q(b(\tau), c^\star)$.

Consider then a solution to the following autonomous first order differential equation:

$$b'(\tau) = H(b(\tau), Q(b(\tau), c^\star)) \tag{10}$$

with initial condition $b(0) = 0$. The debt level $b(\tau)$ strictly increases in the time since last default, and as a result, the associate price that keeps consumption constant, $q(\tau) = Q(b(\tau), c^\star)$, also strictly increases.

Let $b_T \in (0, \overline{b}(c^\star))$ be the debt value such that $Q(b_T, c^\star) = \overline{q}$. That is, b_T is the level of debt such that consumption equals $c^\star$ when the price is $\overline{q}$. Let us define T to be such that $b(T) = b_T$. This value of T is the key threshold: while keeping consumption constant at $c^\star$, T represents the moment where the bond price must reach $\overline{q}$ (the price consistent with a constant default arrival rate equal to δ).

Differentiability of $Q(b, c)$ implies that $q(\tau)$ is differentiable and by the chain rule:

$$q'(\tau) = Q_b(b(\tau), c^\star) H(b(\tau), Q(b(\tau), c^\star)) > 0. \tag{11}$$

From the candidate path of bond prices, $q(\tau)$, we can obtain the corresponding evolution of reputations, $\rho(\tau)$, that are consistent with this evolution of prices. In particular, given the path $q(\tau)$, let the path of *unconditional* arrival rates of default, $x(\tau) \equiv (1 - \rho(\tau)) F'_\tau(\tau)$, satisfy the pricing equation (8), now written as

$$x(\tau) = \frac{q'(\tau) + (i + \lambda)(1 - q(\tau))}{q(\tau)}. \tag{12}$$

which is well defined as $q(\tau) > \underline{q} > 0$ for $\tau \in [0, T]$.

Having thus obtained the path for unconditional arrival rates of default, $x(\tau)$, we can obtain the evolution of the market belief from Bayes' rule, which delivers:

$$\rho'(\tau) = (1 - \rho(\tau))\epsilon + \rho(\tau)(x(\tau) - \delta). \tag{13}$$

with initial condition $\rho(0) = 0$. This is a first-order linear differential equation, thus determining a unique candidate path $\rho(\tau)$ given $x(\tau)$.

Thus, given a candidate $c^\star$, we have constructed candidate equilibrium objects, b, q, ρ for $\tau < T$ where T is such that $\lim_{\tau \to T} q(\tau) = \overline{q}$ and $\lim_{\tau \to T} b(\tau) = b_T$, and both q and b are strictly increasing over the domain.

Construction of $(b(\tau), q(\tau), \rho(\tau))$ for $\tau \geq T$. For $\tau \geq T$, we conjecture that the opportunistic government defaults immediately and thus $\rho(\tau) = 1$ and $q^o(t) = 0$ for $\tau \geq T$. Equations (5) and (6) imply

$$q(\tau) = \overline{q} \text{ for } \tau \geq T \tag{14}$$

The debt evolution then solves:

$$b'(\tau) = H\left(b(\tau), \overline{q}\right) \text{ for } \tau \geq T \tag{15}$$

with initial condition $b(T) = b_T$.

Construction of $\{F_\tau\}_{\tau=0}^{\infty}$. For $\tau < T$, the definition of Markov strategies implies that $F'_\tau(s) = F'_s(s)(1 - F_\tau(s))$ for all $s \in [\tau, T)$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}s} \log(1 - F_\tau(s)) = -\frac{x(s)}{1 - \rho(s)},$$

where we have used that $x(s) = (1 - \rho(s))F'_s(s)$. Using our conjecture that $F_\tau(\tau) = 0$ together with a guess that $\rho(\tau) < 1$ for $\tau \in [0, T)$, we integrate the above and obtain

$$F_\tau(s) = \begin{cases} 1 - \exp\left[-\int_\tau^s \frac{x(\hat{s})}{1-\rho(\hat{s})} d\hat{s}\right] & \text{for } s \in [\tau, T), \\ 1 & \text{for } s \geq T. \end{cases} \tag{16}$$

For $\tau \geq T$, we set $F_\tau(s) = 1$ for all $s \geq \tau$, or that the opportunistic government defaults immediately if it has been weakly longer than T since the last default.

Continuity of Beliefs and Equilibrium. To recap, the assumption of constant consumption c^* allows us to construct a candidate path of bond levels, $b(\tau)$, and prices, $q(\tau)$, which induce this constant level of consumption by the commitment type. These bond prices then imply the unconditional default rates, $x(\tau)$, which justify them. These unconditional default rates then imply the evolution of reputation $\rho(\tau)$, and finally, the unconditional default rates and reputation determine the conditional default rates $F'_\tau(\tau)$.

A difficulty of this construction is that the paths $(b(\tau), q(\tau), \rho(\tau))$ depend on the posited value of $c^\star$. For general posited $c^\star$ values, our resulting constructed $\rho(\tau)$ will not be continuous. We next argue that continuity of our constructed $\rho(\tau)$ is both necessary and sufficient for our construction to be a Markov equilibrium, and that there exists a posited $c^\star > y$ where $\rho(\tau)$ is in fact continuous.

To see why continuity of beliefs is *necessary*, note that in our construction, bond prices $q(\tau)$ are continuous everywhere, including at T. But these constructed prices do not necessarily satisfy Condition 1 of our definition of a Markov equilibrium. In particular, our constructed default behavior requires the break even bond price at $\tau = T$ to be $\overline{q}$, and the the break even bond price at $\tau = T^-$ (the moment before T) to be $\rho(T^-)\overline{q}$. Thus if $\rho(T^-) < \rho(T) = 1$, the break even bond prices implied by default behavior are discontinuous, and thus our continuous constructed bond prices do not satisfy Condition 1. Put simply, if beliefs jump at T, then bond prices must also jump at T, and our constructed bond prices do not allow this.

The following proposition shows that continuity of beliefs is also *sufficient* for our candidate Markov equilibrium to be an equilibrium.

Proposition 1. *For given $c^\star$ such that $y < c^\star < C(0, \frac{i+\lambda}{i+\lambda+\delta})$, let $\{q, \rho, b\}$ be constructed as above. If $\rho(\tau) \in [0, 1)$ for $\tau \in [0, T)$ and $\rho(T^-) = 1$ (or ρ is continuous at T), then $\{F_\tau\}_{\tau=0}^{\infty}$ as constructed above is well defined and together with $\{q, \rho, b\}$ constitute a Markov equilibrium.*

Proof. See Appendix A. □

As Proposition 1 makes clear, for a Markov equilibrium of this type to exist, it is necessary and sufficient that to show there exists $c^\star$ such that ρ remains below 1 before T and is continuous at T, to which we now turn.

Proposition 2. *There exists $c^\star \in (y, C(0, \underline{q}))$ such that the constructed $\{q, \rho, b, \{F_\tau\}_{\tau=0}^{\infty}\}$ exists and is a Markov equilibrium.*

Proof. See Appendix D. □

As we show in the proof of Proposition 2, given that the price is strictly increasing for $\tau \leq T$, $x(\tau) > \delta$ in $[0, T)$, and thus the hazard rate of default conditional on the government being the opportunistic type, $F_\tau'(\tau) = x(\tau)/(1 - \rho(\tau))$, approaches infinity as τ approaches T.

Note that *nowhere* in our construction are any parameters associated with the preferences of the opportunistic type. The preferences of the opportunistic type — its rate of time preference and its utility function (and thus risk aversion) — are not relevant. The reason is that, in equilibrium, it faces no consumption variation either across states of nature (the realizations of arrivals of its Poisson default events) or across time. The preferences of the commitment type do not enter anywhere in the construction either. (In fact, to this point, we haven't even introduced them.) The preferences of the commitment type enter our analysis only when checking whether it will prefer to follow the borrowing rule $H(b, q)$. This, and whether the opportunistic type prefers to follow H, we check for in a later section.

5 An Example

This section presents a computed example to illustrate the nature of our constructed Markov equilibrium. The parameters of our model are the endowment level y, the switching probabilities ϵ and δ, the outside world discount rate i, the coupon debt maturity parameter λ, and the net borrowing function of the commitment type $H(b, q)$ (which together imply the consumption function of the commitment type $C(b, q) = y - (i + \lambda)b + q(H(b, q) + \lambda b)$).

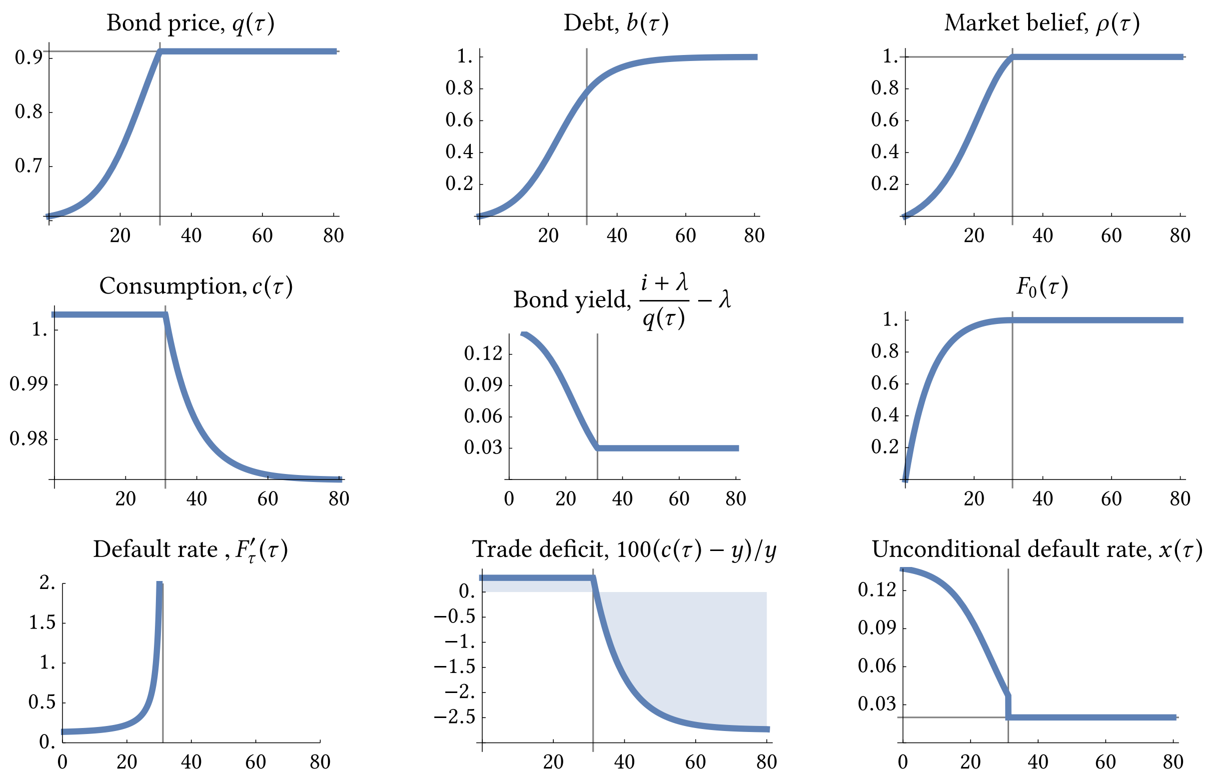

Figure 1: Equilibrium path starting from $\rho_0 = 0$ and $b_0 = 0$. H is as in equation (17). The rest of the parameters are $y = 1$, $\epsilon = 0.1$, $\delta = 0.02$, $i = 0.01$, and $\lambda = 0.2$. The value of T is represented by the vertical line.

For the commitment type's borrowing function $H(b, q)$ and its corresponding consumption function $C(b, q)$, we chose

$$H(b, q) = \max\left\{r^\star - \left(\frac{i + \lambda}{q} - \lambda\right), 0\right\}(y - b)\,. \tag{17}$$

This is similar to the solution to a deterministic optimization problem of a country with log utility and a discount rate of $r^\star$ who believes it can sell debt at the constant bond price q (and thus faces an interest rate of $\frac{i+\lambda}{q} - \lambda$), with the exception that we focus only on borrowing (not saving) andset net borrowing proportional to $(y - b)$, whereas in the deterministic problem, net borrowing is proportional to $(\frac{y}{i} - b)$ where y/i corresponds to the natural debt limit.

Next, we normalize $y = 1$ and choose our other parameters relative to a unit of time being one year. Thus, if we set $\epsilon = 0.01$ and $\delta = 0.02$, this implies a 1% chance that an opportunistic government dies in the next year to be replaced by a commitment government, and a 2% chance that a commitment government dies to be replaced by an opportunistic government. (And thus, the country has a commitment government one-third of the time.) We set the outside world discount rate $i = 0.01$ and $\lambda = 0.2$, corresponding to a yearly principal payoff of 20% or roughly five-year debt. These imply that in the long run (after date T is reached and thus the government is certainly the commitment type), the probability of default is 2% per year (from $\delta = 0.02$), and thus the long-run interest rate is 3% (from $i + \delta = 0.03$) and the long-run bond price is $0.913 = \frac{i+\lambda}{i+\lambda+\delta}$ (as opposed to a bond price of one if lending were riskless). Finally, we set $r^\star = 0.15$. Under these parameters, equation (17) satisfies Assumption 1 with $\overline{B} = y$ (see Appendix E), and thus an equilibrium exists.

Figure 1 displays some relevant time paths for these parameters (again, where all paths start over given a default) in the constructed equilibrium.[17] Here, it takes about 31 years for the market belief that the government is a commitment type to go from $\rho = 0$ to $\rho = 1$. In this time, debt goes from $b = 0$ to $b = 0.8$ (or a debt/GDP ratio of 80%) to, eventually, $b = 1$ (or a debt/GDP ratio of 100%), whereas the bond price goes from 0.6 to its long run value of 0.91. Consumption stays steady at about 0.3% above endowment for these 31 years, and then smoothly decreases over the next 30 years to about 97% of the country's endowment. The country's default rate starts at about 14% per year, decreasing to, eventually, 2% per year.

A useful consequence of analytically characterizing a Markov equilibrium is that nearly every moment of an example can be calculated as opposed to simulated. Here, for instance, once a country's interest rate reaches its long-run value $T = 31$ years after its last default, the expected time to default is $\int_0^\infty t\delta e^{-\delta t} dt = \frac{1}{\delta}$, or for these example parameters, 50 years. The average length of time to graduate after a default (call this m) is a more difficult formula, but can be expressed as

$$m = T + \int_0^T t x(t) e^{\int_t^T x(s) ds} dt,$$

which for these example parameters is a bit less than 200 years. (Recall it takes $T = 31$ years in this example go from default to graduation *conditional on not defaulting*. The probability of not defaulting for T years after a default, however, is $e^{-\int_0^T x(t) dt}$, which for these parameters is a bit less than 10%.)

[17] Numerically, we solve for $b(\tau), q(\tau)$ jointly by solving $b'(\tau) = H(b(\tau), q(\tau))$, $q'(\tau) = -\frac{C_b(b(\tau),q(\tau))}{C_q(b(\tau),q(\tau))} H(b(\tau), q(\tau))$ with initial conditions $b_0 = 0$ and $q(0) = q_0$ where $C(0, q_0) = c^\star$; and iterate on $c^\star$ until $\rho(T) = 1$. Although we do not have a proof of uniqueness, we only find one such $c^\star$.

6 Characterizing All Markov Equilibria

In this section, we give a tight characterization of *all* Markov equilibria such that both types follow *any* borrowing rule $H(b,q)$ satisfying Assumption 1, and show all are of the type constructed in the previous section.

Specifically, we show in *any* Markov equilibrium starting from $(b=0, \rho=0)$, that the continuation value to the opportunistic government equals a constant. (And thus, there is no value, on the equilibrium path, to having a good reputation.) This then implies that the on-path consumption of the opportunistic type must also be constant. Finally, we show that if this constant on-path consumption exceeds the country's endowment (as it does for all of our computed equilibria), then there exists a date $T > 0$ such that $F_\tau(\tau) = 1$ for all $\tau \geq T$, and where the price path $q(\tau)$ is continuous at T. These two characteristics (constant on-path consumption, $c^\star$, and a date T such that $F_\tau(\tau) = 1$ for all $\tau \geq T$ with bond prices continuous at T) are all we used in the previous section to construct our candidate equilibrium. Everything else about the constructed equilibrium was implied by the equilibrium conditions. Thus *all* Markov equilibria with $c^\star > y$ have the form of our constructed equilibrium from the previous section.

We now turn to proving these characterizations. For a given Markov equilibrium $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$, let $V(\tau)$ and $c(\tau)$ denote the associated value to the opportunistic government and consumption level as a function of time since the last default, τ. We first establish that $V(0) \geq \frac{u(y)}{r+\epsilon}$ and that for all $\tau \geq 0$, $V(\tau) = V(0)$.

Lemma 1. *If $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$ is a Markov equilibrium with associated value $V(\tau)$, $V(0) \geq \frac{u(y)}{r+\epsilon}$.*

Proof. See Appendix F. □

Proposition 3. *If $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$ is a Markov equilibrium with associated value $V(\tau)$, for all $\tau \geq 0$, $V(\tau) = V(0)$.*

Proof. See Appendix G. □

The intuition behind the *necessity* of a constant value function is as follows: First, the continuation value can never fall below its initial value, otherwise an opportunistic government could default immediately and get the initial value. Next, eventually debt becomes high enough so that the value after a sufficiently long period of no default must also equal its initial value due to the opportunistic government certainly defaulting. Thus if a continuous value function is ever above its initial value, it must do so by rising and then falling over some interval. In this interval where the value is above its initial value, the opportunistic type must be not defaulting with certainty, while the time to when it starts defaulting again moves closer, so the bond price must be falling.

This and that debt keeps accumulating in this interval implies consumption must be falling over time. But this contradicts the value function ever rising.

From this proposition, and the two results we present now (Lemma 2 and Proposition 4), it follows that the equilibrium consumption path for the opportunistic type must be constant:

Lemma 2. *If* $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$ *is a Markov equilibrium with associated constant value* V*, then for any* $\tau \geq 0$ *and* $\Delta > 0$ *such that* $F_\tau(\tau+\Delta) < 1$*,* $c(t) = c^\star \equiv u^{-1}\big((r+\epsilon)V\big)$ *for almost all* $t \in [\tau, \tau+\Delta]$.

Proof. See Appendix H. □

Consider now an equilibrium where $c^\star > y$. Define

$$T \equiv \inf\{s \geq 0 | F_0(s) = 1\}. \tag{18}$$

Note that $c^\star > y$ guarantees that $T > 0$.[18] In the following proposition, we show that after T, default always occurs immediately if the opportunistic type is in government.

Proposition 4. *Let* $(\{F_\tau\}_{\tau=0}^{\infty}, q(\tau), \rho(\tau), b(\tau))$ *be a Markov equilibrium with associated constant value* $V > u(y)/(r+\epsilon)$*. Then,* $F_\tau(\tau) = 1$ *for all* $\tau \geq T$ *as defined by* (18).

Proof. See Appendix I. □

Finally, we now argue that the price $q(\tau)$ is continuous at T. The proposition above implies $q(T) = \overline{q}$ and $q(T^-) = \rho(T^-)\overline{q}$ since $q^o(T) = 0$. Since $\rho(T^-) \leq 1$, $q(T) \geq q(T^-)$. Suppose that $q(T) > q(T^-)$. Then consumption after T must be strictly larger than the consumption before T, as the price has strictly increased while debt evolves continuously. This means that, after T, consumption remains higher than $c^\star$, violating the optimality of the default at T.

To recap, we have shown all Markov equilibria exhibit constant on-path consumption, a date T such that $F_\tau(\tau) = 1$ for all $\tau \geq T$, and bond prices continuous at T. These three characteristics are sufficient to show all Markov equilibria have the form of our constructed equilibrium.

7 Starting Points Other than $\rho_0 = 0$ and $b_0 = 0$

To this point, we have assumed our game starts with $\rho_0 = 0$ and $b_0 = 0$. This is the relevant subgame after the first default and any subsequent defaults. We now turn to characterizing Markov equilibria for starting values of (b, ρ) other than $(0, 0)$. In Section 8, we use these results to establish that following borrowing rule H is indeed optimal for the opportunistic type (or that the

[18] The previous lemma guarantees that if $T = \infty$, consumption of both types equals (a.e.) $c^\star > y$, which violates the bounded debt restriction.

opportunistic type chooses to reveal its type only by defaulting) and is optimal for the commitment type as long as it is sufficiently impatient.

As discussed in Section 2.1, for the subgame starting from $(b = 0, \rho = 0)$, it is sufficient to collapse (b, ρ) into a single state variable, time-since-last-default, τ. For starting points other than $(b = 0, \rho = 0)$, we need to make strategies a function of (b, ρ) directly, but only before the first default.

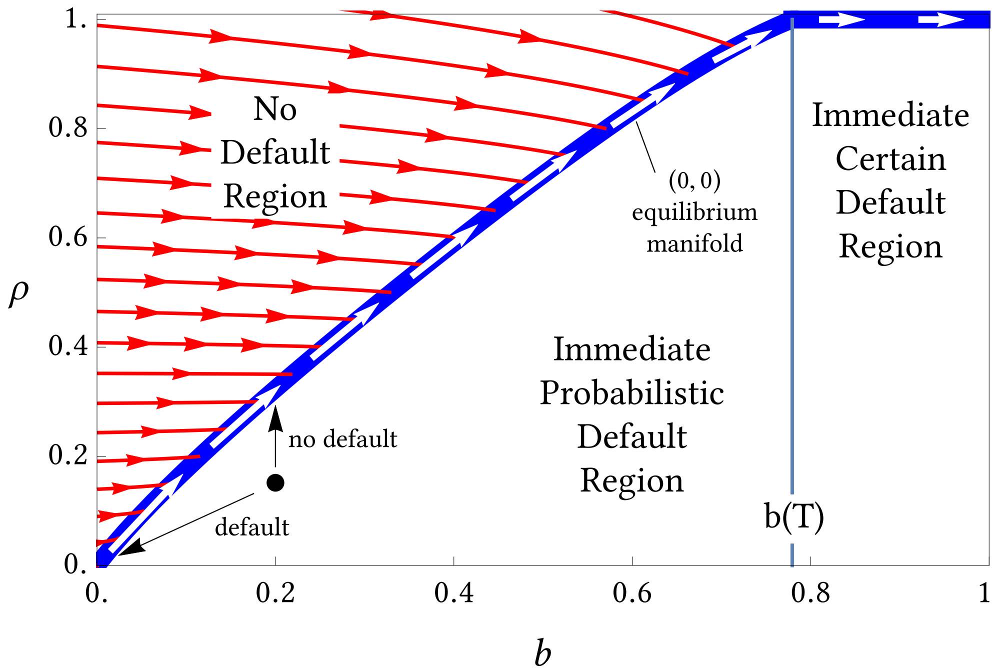

Figure 2: *Default regions. The thick blue line depicts the equilibrium manifold starting from* $b_0 = 0$ *and* $\rho_0 = 0$*. At any starting point above the manifold, no default occurs, and* (b, ρ) *moves along a red line until reaching the manifold. At any starting point below the blue line, the equilibrium jumps to the blue line if no immediate default occurs. To the right of* $b(T)$*, there is immediate certain default by the opportunistic type, independently of* ρ*. To the left of* $b(T)$*, immediate default occurs probabilistically.*

Consider Figure 2. In the $(0, 0)$ game, the state variables (b, ρ) start at $(0, 0)$ and over time (if no default) move to the northeast along the thick blue line (the $(0, 0)$ equilibrium manifold) with both debt and reputation increasing until debt reaches its date T level, $b(T)$, in which case reputation is at its maximum (one), but debt continues to increase until reaching its steady state. (Every time there is a default, the state variables return to $(0, 0)$.)

Next, partition (b, ρ) space into starting values (b_0, ρ_0) such that the government's initial reputation ρ_0 is *equal to* (on the blue line), *greater than* (above the blue line), and *less than* (below the blue line) what its reputation is for that level of debt in the $(0, 0)$ subgame. More formally, note first that since $b(\tau)$ in the $(0, 0)$ subgame is strictly increasing over time, there is a one-to-one mapping between debt levels b and times since the last default τ, and thus we can define $\tau^*(b)$ to be the amount of time it takes to reach debt b in the $(0,0)$ subgame. Since any Markov equilibrium of the $(0,0)$ subgame defines functions $b(\tau)$ and $\rho(\tau)$, the equilibrium manifold is then represented by the function $\rho(\tau^*(b))$.

Next, consider (b_0, ρ_0) such that $\rho_0 = \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is exactly what it is in the $(0,0)$ subgame when it has debt b_0 (or (b_0, ρ_0) is on the blue line). In these cases, assume the opportunistic type follows its strategy from the $(0,0)$ subgame, but starting as if it has been $\tau^*(b_0)$ periods since the last default. Since following this strategy is optimal in the $(0,0)$ subgame, it is optimal starting from (b_0, ρ_0).

Next, assume $\rho_0 > \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is strictly greater than what it is in the $(0,0)$ subgame when it has debt b_0 (or (b_0, ρ_0) is above the blue line). Here, we propose the opportunistic type sets $F_0(t) = 0$ for a specific amount of time t^* (which depends on (b_0, ρ_0)). Its reputation ρ at date $t < t^*(b_0, \rho_0)$ is then

$$\hat{\rho}(t) = \frac{\epsilon}{\epsilon + \delta} + e^{-(\epsilon+\delta)t}\left(\rho_0 - \frac{\epsilon}{\epsilon + \delta}\right),$$

which converges continuously to $\frac{\epsilon}{\epsilon+\delta}$. Since in the $(0,0)$ subgame, $\rho(\tau)$ moves continuously from zero to one, this ensures there exists (t^*, τ) such that $\hat{\rho}(t^*) = \rho(\tau)$. From date t^* on then, we propose the opportunistic type follows the $(0,0)$ equilibrium starting as if it had been τ periods since the last default. Graphically, the state variables (b, ρ) move continuously to the east along the red line associated with (b_0, ρ_0) until hitting the blue line, where from there the game follows the $(0,0)$ equilibrium as if it has been τ periods since a default. Since from time zero to time t, the bond price q is strictly greater than the bond price for that level of debt in the $(0,0)$ equilibrium (since the probability of default is zero for some time), consumption along this path is strictly greater than the consumption of the opportunistic type in the $(0,0)$ subgame, ensuring the opportunistic type is willing to set $F_0(t) = 0$ for $t < t^*$.

Finally, suppose $\rho_0 < \rho(\tau^*(b_0))$, or that the government's initial reputation ρ_0 is strictly less than what it is in the $(0,0)$ subgame when it has debt b_0. Here, if $\rho_0 > 0$ and $b_0 < b(T)$ (labeled the "Immediate Probabilistic Default Region" in Figure 2), we propose the opportunistic type immediately defaults with probability γ such that

$$\rho(\tau^*(b_0)) = \frac{\rho_0}{\rho_0 + (1-\rho_0)(1-\gamma)}.$$

Since $0 < \rho_0 < \rho(\tau^*(b_0))$, then $\gamma \in (0,1]$, and this default behavior ensures that (b, ρ) jumps either to $(0,0)$ (in the case of immediate default) or $(b_0, \rho(\tau^*(b_0)))$ (in the case of no immediate default). The opportunistic type then follows the strategy from the $(0,0)$ game starting from either $\tau = 0$ or $\tau = \tau^*(b_0)$ depending on whether it immediately defaulted. It is willing to set γ

between zero and one because its continuation value is the same in either case.[19] If $\rho_0 > 0$ and $b_0 \geq b(T)$ (labeled the "Immediate Certain Default Region" in Figure 2), we propose the opportunistic government immediately defaults with probability one. Under this strategy, reputation ρ immediately jumps to one if no default. Here, the opportunistic government strictly prefers to default.

If $\rho_0 = 0$, we propose the opportunistic type immediately defaults with probability one. Here, Bayes' rule doesn't apply for calculating beliefs conditional on not defaulting; thus we are free to set the belief. If $b_0 \leq b(T)$, setting ρ conditional on no default to $\rho(\tau^*(b_0))$ again ensures the opportunistic type's payoff is the same regardless of whether he defaults or not and is thus willing to set γ to one.[20] If $b_0 > b(T)$, the opportunistic type finds it strictly optimal to default.

8 Endogenous Debt Issuances

In this section we allow for strategic debt issuances. That is, to this point, we have assumed both government types follow the debt issuance rule $H(b, q)$ and the only strategic behavior allowed is for the opportunistic government to default. Among other things, such an assumption rules out the possibility of the commitment type actively trying to separate from the opportunistic type through its debt issuance behavior.

To this end, we consider two separate questions: First, for our constructed equilibrium, will both government types be willing to follow the borrowing rule H if allowed to deviate when issuing debt? (That is, does our constructed equilibrium continue to be an equilibrium if debt issuance becomes a strategic choice by both government types?) Second, are there any *separating equilibria* where the two government types reveal themselves through their debt issuance behavior? Subject to conditions we outline below, we show here that our constructed equilibrium continues to be an equilibrium when allowing for endogenous debt issuances, and that no Markov separating equilibria where the two government types reveal themselves through their debt issuance behavior exist. We find it useful to answer the second question first.

[19] Note that if the strategy calls for γ high enough that ρ jumps above $\rho(\tau^*(b_0))$ if no default, the opportunistic government will find it optimal to deviate and set $\gamma = 0$. If the strategy calls for γ low enough so that ρ fails to reach $\rho(\tau^*(b_0))$ if no default, the strategy calls for another immediate probabilistic default to reach $\rho(\tau^*(b_0))$, which is the same thing as choosing γ such that $\rho(\tau^*(b_0))$ is reached immediately if no default. Any strategy other than immediate probabilistic default would imply lower consumption than $c^\star$ for some amount of time, thus inducing immediate default, unless bond prices are *higher* at the lower reputation and same debt as when on the manifold — a property not compatible with equilibrium.

[20] Again note that if we set ρ after no default greater than $\rho(\tau^*(b_0))$, the opportunistic type will find it optimal to deviate and not default, and if we set ρ after no default less than $\rho(\tau^*(b_0))$, the strategy again calls for another immediate (probabilistic) default.

8.1 On The Lack Of Separating Markov Equilibria

Here, we establish that our continuous time game has no Markov equilibria where a commitment type issues positive new bonds and the opportunistic type issues a different level of new bonds, subject to one condition: such an equilibrium must be the limit of equilibria of the discrete time version of our game as $\overline{\Delta} \to 0$, where the value to the opportunistic type, $V(b, \rho|\overline{\Delta})$ is strictly decreasing in b – a property our constructed equilibrium *does* have. Recall in the discrete move game timing, debt issuances and coupon payments occur just *after* clock ticks $t \in \{0, \overline{\Delta}, 2\overline{\Delta}, \ldots\}$, then default occurs or not, and then a type switch occurs or not (with probabilities $\epsilon\overline{\Delta}$ and $\delta\overline{\Delta}$ respectively). Let $q(b^+, \rho|\overline{\Delta})$ denote the price of a bond promising to pay a stream $((i+\lambda)\overline{\Delta}, (i+\lambda)\overline{\Delta}(1-\lambda\overline{\Delta}), (i+\lambda)\overline{\Delta}(1-\lambda\overline{\Delta})^2, \ldots\}$ starting in $\overline{\Delta}$ units of time, when new debt and reputation are equal to (b^+, ρ) at the time the bond is issued.

Consider an opportunistic government with debt b that reveals itself by choosing a different b^+ than what the strategy calls for the commitment type to issue. Then its lifetime payoff is

$$u\left(y - (i+\lambda)b + q(b^+, 0|\overline{\Delta})\frac{b^+ - (1-\lambda\overline{\Delta})b)}{\overline{\Delta}}\right)\overline{\Delta} + \frac{1-\epsilon\overline{\Delta}}{1+r\overline{\Delta}}\max\left\{V(0, \epsilon\overline{\Delta}|\overline{\Delta}), V(b^+, \epsilon\overline{\Delta}|\overline{\Delta})\right\}.$$

where the max in the last term represents the subsequent default decision. Note that since we restrict our attention to equilibria such that $V(b, \rho|\overline{\Delta})$ is strictly decreasing in b, any equilibrium strategy must call for certain default by the opportunistic type after issuing $b^+ > 0$. Thus $q(b^+, 0|\overline{\Delta}) = 0$ for any $b^+ > 0$ different from the level a proposed strategy calls for the commitment type to issue, and its consumption is $y - (i+\lambda)b$. That is, the choice for an opportunistic government is always whether to raise zero revenue from new bond issuances or mimic.

Suppose a proposed strategy calls for the commitment type to issue positive new debt and the opportunistic type not to mimic. Then the opportunistic government's current consumption is $y-(i+\lambda)b$ and its continuation payoff is $V(0, \epsilon\overline{\Delta})$ (since it has revealed its type by not mimicking). If instead the opportunistic type mimics and then defaults, its consumption is $y - (i+\lambda)b$ plus the revenue raised from the debt issuance (which is positive since $\rho = 1$ if the opportunistic type deviates and mimics) and its continuation payoff is again $V(0, \epsilon\overline{\Delta})$, which dominates not mimicking. Thus whenever a strategy calls for commitment type to raise revenue by issuing new debt, it must call for the opportunistic type to mimic.

8.2 Optimality of following $H(b, q)$

Given that no Markov equilibria exist where types reveal themselves through their debt issuances, there is still the question of whether our constructed equilibrium continues to be an equilibrium when debt issuance is a strategic choice. Here, we rely on our characterization of play for arbitrary

(b, ρ) starting points. Again, we assume the borrowing country is supposed to follow H whenever $\rho \geq \rho(\tau^*(b))$ (which occurs with equality in the $(0, 0)$ subgame and as a strict inequality for sufficiently high ρ_0 relative to b_0). Further, we assume that outside lenders believe $\rho = 0$ whenever a country attempts to borrow differently (and discuss this assumption at the end of this section).

First consider the opportunistic type. In any Markov equilibrium, its value is the same for all (b_0, ρ_0) such that $\rho_0 \leq \rho(\tau^*(b_0))$ and strictly above this common value when $\rho_0 > \rho(\tau^*(b_0))$. If it deviates and borrows differently from $H(b, q)$, its reputation ρ becomes zero, and its value is either unchanged (in the case where $\rho_0 \leq \rho(\tau^*(b_0))$) or declines (in the case where $\rho_0 > \rho(\tau^*(b_0))$). Thus, the opportunistic type will always find it weakly optimal to follow the borrowing rule H, choosing to reveal its type only by defaulting (and where in the discrete time game, the opportunistic type finds it strictly optimal to follow the borrowing rule as long as its reputation falls at all when deviating, even if not all the way to zero.)

Next, consider the commitment type. We now allow for the commitment type to choose its issuances, but impose *the restriction that it cannot buy back the bonds.* Since we have assumed that if a country attempts to borrow differently than H, it is assumed to be certainly the opportunistic type, and the opportunistic type is supposed to default immediately with probability one whenever $\rho = 0$ and $b > 0$, this implies a bond price of zero immediately after a country fails to follow H (which is the same as not being able to borrow).[21] Thus, consider a commitment type that deviates from H for an interval of time $[t, t + \Delta]$. During this period, ρ and q remain at 0, and the commitment type must simply consume its endowment while making its coupon payments, reducing its debt at the rate λ, a result that follows from our no buyback restriction.

Suppose then at any point in the game where it has debt $b \geq 0$, a commitment type with bounded utility function u^c and discount factor r^c considers using this strategy for an amount of time Δ. At the end of this period, the level of debt is $\hat{b} = e^{-\lambda\Delta} b$. Its value from following this strategy (as a function of its current debt, b, and Δ) is

$$\hat{V}^c(b, \Delta) \equiv \int_0^\Delta e^{-r^c s} u^c(y - (i + \lambda)e^{-\lambda s} b) ds + e^{-r^c \Delta} V^c(e^{-\lambda\Delta} b),$$

where $V^c(b) = \int_{\tau^*(b)}^\infty e^{-r^c(s - \tau^*(b))} u(c(s)) ds$, or $V^c(b)$ is the value to the commitment type of following the equilibrium as if it has been $\tau^*(b)$ periods since the last default. Here, the first integral is the commitment type's payoff between the date it starts to deviate and the date it stops, and the second integral is its payoff from then on. At this date, since it starts following the rule H and doesn't default (since it can't), its reputation jumps to $\rho(\tau^*(\hat{b}))$, and it is back on the equilibrium

[21] We have confirmed this property in the discrete time version of this game as well. Under the assumption that outside lenders believe borrowing differently than H implies the borrower is the opportunistic type, in any Markov perfect equilibrium, the price of such debt is zero.

path as if it has been $\tau^*(\hat{b})$ periods since the last default. The derivative of $\hat{V}^c$ with respect to Δ is

$$\frac{\partial \hat{V}^c(b,\Delta)}{\partial \Delta} = e^{-r^c\Delta}\left\{u\left(y-(i+\lambda)\hat{b}\right) - r^c V^c\left(\hat{b}\right) - \lambda \hat{b}\frac{dV^c\left(\hat{b}\right)}{d\hat{b}}\right\}$$

That $\lim_{r^c\to\infty} r^c V^c(\hat{b}) = u(c(\tau^*(\hat{b})))$ implies $\lim_{r^c\to\infty} \frac{dV^c(\hat{b})}{d\hat{b}} = 0$ as $\frac{dV^c(\hat{b})}{d\hat{b}} = \tau^{*\prime}(\hat{b})[r^c V^c(\hat{b}) - u(c(\tau^*(\hat{b}))]$. Given that debt is strictly increasing in the path of the constructed equilibrium, we have that $u(y-(i+\lambda)\hat{b}) < u(c(\tau^*(\hat{b})))$. Hence, there exists r^c sufficiently high such $\frac{\partial \hat{V}^c(b,\Delta)}{\partial \Delta} \leq 0$ for any $\Delta > 0$, and the commitment type is thus willing to follow the constructed equilibrium strategy for any $b \geq 0$.

9 Conclusion

In this paper, we presented a tractable sovereign debt model where the borrower's reputation and its interaction with default events generate dynamics of debt and asset prices that are consistent with several facts.

In our model, a government that defaults loses its reputation, and it takes periods of borrowing and not defaulting to eventually restore it. During these periods, bond prices are low and default frequencies are high, as in the data. Further, relative to countries that have not recently defaulted, debt levels are low. In fact, in our model, as in the data, countries with low debt levels face relatively high interest rates, a phenomenon referred to as "debt intolerance." In our model, a country can "graduate" into the set of "debt-tolerant" countries by not defaulting for a sufficiently long period of time, as perhaps Mexico has done by not defaulting since the 1980s.

In the data, default is less than fully predictable and somewhat untied to fundamentals. Recent work has emphasized this fact as an argument for introducing features that lead to multiple equilibria in the standard sovereign debt model. In our environment, such an outcome arises naturally. Equilibrium default in our model is necessarily random. Such randomness is a fundamental ingredient for the dynamics of learning and reputation.

A Proof of Proposition 1

Let us first state two preliminary results:

Lemma 3. *Give a constant price, q, the differential equation $b'(\tau) = H(b(\tau), q)$ with initial condition $b(0) = b_0 \in [0, \overline{B}]$ has a unique solution defined for all $\tau \geq 0$ and where $b(\tau)$ converges monotonically to a steady state.*

Proof. Assumption 1 part (i) guarantees that the differential equation $b'(\tau) = H(b(\tau), q)$ with $b(0) = b_0$ has a unique (local) solution. The role of Lipschitz continuity can be found in any ordinary differential equations book. See for example Hale (2009) Theorem 3.1. As it is standard, uniqueness in a one-dimensional autonomous system implies that the dynamics of $b(\tau)$ are monotone. Otherwise $b'(\tau)$ must cross zero in some finite time t_1, and the stationary solution that keeps $b(\tau)$ constant at $b(t_1)$ is also a solution to the ODE, violating uniqueness. Finally, Parts (iv) and (v) of Assumption 1 imply that a local solution can be extended for $\tau \geq 0$, as the solution $b(\tau)$ does not exit the range $[0, \overline{B}]$. $\square$

Next, we show that the function $Q(b, c)$ is differentiable. Let $\mathbb{Y} \equiv \{(b, c) | y < c < C(0, 1) \text{ and } 0 < b < \overline{b}(c)\}$.

Lemma 4. *The function $Q(b, c)$ is differentiable in $\mathbb{Y}$ with $Q_b > 0$ and $Q_c > 0$.*

Proof. We have already argued in the text that $Q(b, c) \in (\underline{q}, 1]$ exists for any and $c \in (y, C(0, 1))$ and $b \in [0, \overline{b}(c)]$ The definition of Q is:

$$c = y - (i + \lambda)b + Q(b, c)(H(b, Q(b, c)) + \lambda b)$$

Note that $c > y$, implies that $H > 0$, and thus H is differentiable with respect to both arguments given Assumption 1 part (vi). The implicit function theorem then delivers that:

$$Q_b(b, c) = \frac{i + \lambda(1 - Q) - QH_b(b, Q)}{H(b, Q) + \lambda b + QH_q(b, Q)} \text{ where } Q = Q(b, c)$$

$H_q \geq 0$ from Assumption 1 part (iii), and thus the denominator is strictly positive given $H > 0$. We also know that $H_b \leq 0$ from Assumption 1 part (i), and thus the numerator is strictly positive, so $Q_b(b, c) > 0$.

Similarly,

$$Q_c(b, c) = \frac{1}{H(b, Q(b, c)) + \lambda b + Q(b, c)H_q(b, Q(b, c))}$$

which shares the same strictly positive denominator, and thus $Q_c(b, c) > 0$. $\square$

We are now ready to state the proof.

Proof of Proposition 1. Suppose that we have b, q, and ρ as constructed with an associated T. We first argue that $\{F_\tau\}_{\tau=0}^{\infty}$ can be constructed if $\rho(\tau) < 1$ for $\tau < T$. To see this note that (12) implies that $x(\tau)$ is continuous as q is differentiable as shown in equation (11). Hence $x(\tau)/(1-\rho(\tau))$ is well defined and continuous for $\tau \in [0, T)$. Thus it is integrable in a compact set, and $F_\tau(s)$ in equation (16) is well defined for all $\tau \in [0, T)$ and $s \in [\tau, T)$.

We now proceed to argue that if $\rho(\tau)$ is continuous at T, the constructed candidate functions $\{F_\tau\}_{t=0}^{\infty}$, $\{q(\tau), \rho(\tau), b(\tau)\}$ satisfy the five conditions for equilibrium stated in Definition 2.

Condition 3 (debt evolution): holds by construction, given that (10) and (15) hold. Note that for the latter, Lemma 3 guarantees that $b(\tau)$ remains in $[0, \overline{B}]$ for all $\tau \geq T$.

Condition 4 (opportunistic type optimizes): first we show that $c(\tau)$ is continuous at all τ and weakly decreasing for $\tau \geq T$. Continuity follows from the fact that both $b(\tau)$ and $q(\tau)$ are continuous, and thus $c(\tau) = C(b(\tau), q(\tau))$ is continuous. For $\tau < T$, $c(\tau)$ is constant. For $\tau > T$, we have that $H(b(\tau), q(\tau)) \geq 0$ as $H(b(T), q(T)) > 0$ and convergence is monotone from time T onwards with a constant price by Lemma 3. Given that $b(\tau)$ is weakly increasing, it follows that $c(\tau)$ is weakly decreasing by Assumption 1(ii) and $q(\tau) < 1$.

That $c(\tau)$ is continuous for all τ, constant for all $\tau < T$ (by construction), and weakly decreasing for $\tau > T$ implies that the conjectured default strategy for the opportunistic type is optimal. That is, the opportunistic type consumes $c^\star$ at all times, and as a result it is indifferent between defaulting or not before $\tau \leq T$ and (weakly) prefers to default for $\tau > T$, thus ensuring that Condition 4 is satisfied.

Condition 2 (market beliefs are rational): For $\tau \geq T$, $\rho(\tau) = 1$ and $F_\tau(\tau) = 1$, and immediately equation (3) is satisfied for $\tau > T$. For $\tau = T$, equation (3) also holds if $\rho(T^-) > 0$ and does not apply if $\rho(T^-) = 0$.

For $\tau < T$, by construction $\rho'(\tau) = (1-\rho(\tau))\epsilon + \rho(\tau)(x(\tau) - \delta)$, ensuring Bayes' rule holds locally given the conjectured default behavior $F'_\tau(\tau) = x(\tau)/(1-\rho(\tau))$.

Condition 1. (Foreign investors break even in equilibrium.): In our construction, we derive the *unconditional* bond price $q(\tau)$, but our definition of a Markov equilibrium is in terms of *conditional* prices $q^c(\tau)$ and $q^o(\tau)$.

The existence of q^o and q^c, given $\{F_\tau\}_{\tau=0}^{\infty}$, follows from a contraction mapping developed in Appendix B.

We next need to show that the price generated from q^o and q^c given equation (5) using the market beliefs ρ coincides with q. Using the continuity of ρ, let $\hat{q}$ be defined as the price consistent with q^o, q^c and beliefs ρ:

$$\hat{q}(\tau) \equiv \rho(\tau)q^c(\tau) + (1 - \rho(\tau))q^o(\tau).$$

Continuity of q^o and q^c (shown in Appendixes B and C), together with continuity of ρ, guarantees that $\hat{q}$ is continuous.

Note that $\hat{q}(\tau) = q^c(\tau) = \frac{i+\lambda}{i+\lambda+\delta} = q(\tau)$ for $\tau \geq T$, where the second equality follows from the proposed F_τ. Taking derivatives of the integral forms (6) and (7), together with (4) and the definition of $x(\tau)$, implies that $\hat{q}$ satisfies

$$\hat{q}'(\tau) = (i + \lambda + x(\tau))\hat{q}(\tau) - (i + \lambda) \tag{19}$$

for $\tau < T$. Note that differentiability of Q with respect to b (Lemma 4) and the continuity of H (Assumption 1vi) imply that $\lim_{\tau \to T^-} q'(\tau)$ is finite and we let $x(T) = \lim_{\tau \to T^-} x(\tau) \in [\delta, \infty)$. Using this extension of x, it follows that $\hat{q}$ in $[0, T]$ solves an initial value problem (IVP): it satisfies equation (19) with boundary condition $\hat{q}(T) = q(T)$. This IVP is a first-order linear ordinary differential equation on $[0, T]$ with time dependent and continuous coefficients. Hence, it has a unique solution in $[0, T]$. Given that q solves the same IVP, it follows that q and $\hat{q}$ are the same.

Condition 5 (Each F_τ is a cumulative distribution function and they are consistent with each other.): First we need to show that each $F_\tau \in \Gamma$. For $\tau < T$, the function defined in equation (16) lies in $[0, 1]$ and is differentiable and increasing, given that $x(s) \geq 0$ and $\rho(s) < 1$ for $s < T$ (from the arguments in the proof of Condition 2 above). Given that $F_\tau(s) = 1$ for $s > T$, it follows that $F_\tau \in \Gamma$. For $\tau \geq T$, $F_\tau \in \Gamma$ by construction.

Next we argue that equation 2 holds. To see this, note that the equation holds for any $s \geq T$, as $F_\tau(s) = F_m(s) = 1$ for all τ, m. And for $s < T$, the equation holds given the exponential form in (16).

All the conditions for the definition of equilibrium hold. $\square$

B Existence and uniqueness of a q^o and a continuous q^c

Here we show that given $\{F_\tau\}_{\tau=0}^{\infty}$, there exists a unique q^o and a unique continuous q^c that satisfy the integral equations described in the main body of the paper.

Instead of working with vector-valued operators, the idea of the proof is to substitute the equation for q^o into q^c. Then, to prove the existence, uniqueness and continuity of q^c, we construct a contraction T mapping the space of bounded, continuous functions to itself and where q^c is a fixed point of this mapping.

First, define $T^o\{f\}(\tau)$ as

$$T^o\{f\}(\tau) = \int_0^\infty \left[\left(\int_0^s (i+\lambda)e^{-(i+\lambda)\tilde{s}}\mathrm{d}\tilde{s} + e^{-(i+\lambda)s} f(\tau+s) \right) (1 - F_\tau(\tau+s)) + \int_0^s \left(\int_0^{\tilde{s}} (i+\lambda)e^{-(i+\lambda)\Delta}\mathrm{d}\Delta \right) \mathrm{d}F_\tau(\tau+\tilde{s}) \right] \epsilon e^{-\epsilon s}\mathrm{d}s. \quad (20)$$

In words, $T^o\{f\}(\tau)$ is the value of a bond given an opportunistic government where upon a type switch, the owner receives an arbitrary payoff $f(\cdot) \in [0,1]$.

Next likewise, define $T^c\{g\}(\tau)$ as

$$T^c\{g\}(\tau) = \frac{i+\lambda}{i+\lambda+\delta} + \int_0^\infty e^{-(i+\lambda+\delta)s} g(\tau+s)\delta\mathrm{d}s. \quad (21)$$

In words, $T^c\{g\}(\tau)$ is the value of a bond given a commitment type government where upon a type switch, the owner receives an arbitrary payoff $g(\cdot) \in [0,1]$.

Finally, let $T\{f\}(\tau) \equiv T^c\{T^o\{f\}\}(\tau)$. Here, T is the value of a bond given a commitment type government where upon two type switches (from commitment to opportunistic and back again), the owner receives an arbitrary payoff $f(\cdot) \in [0,1]$.

We now proceed to showing that T^o and T^c are each well defined, and that T is a contraction on the space of bounded continuous functions. First, we can rewrite T^c and T^o as:

$$T^c\{g\}(\tau) = \underline{q} + \delta H_0(-\tau) \int_\tau^\infty H_0(s)g(s)ds$$
$$T^o\{f\}(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} (1 - F_s(s+\hat{s}))e^{-\epsilon\tilde{s}} dH_1(\hat{s})d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s}) f(s+\tilde{s})(1 - F_s(s+\tilde{s}))d\tilde{s}$$

where

$$\underline{q} = \frac{i+\lambda}{i+\lambda+\delta}, H_0(s) = e^{-(i+\lambda+\delta)s}, H_1(s) = \left(1 - e^{-(i+\lambda)s}\right), H_2(s) = e^{-(i+\lambda+\epsilon)s}$$

and where we used integration by parts to rewrite T^o.

Plugging the equation for T^o back into T^c we obtain that q^c is a fixed point of the operator, T, now written as:

$$T\{f\}(\tau) = g_0(\tau) + \delta\epsilon H_0(-\tau) \int_\tau^\infty \int_0^\infty g_1(s,\tilde{s}) d\tilde{s} ds$$
$$\text{where } g_1(s,\tilde{s}) = H_0(s) H_2(\tilde{s})(1 - F_s(s+\tilde{s})) f(s+\tilde{s})$$

and where

$$g_0(\tau) = \underline{q} + \delta\epsilon H_0(-\tau) \int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s) e^{-\epsilon\tilde{s}} (1 - F_s(s+\hat{s})) dH_1(\hat{s}) d\tilde{s} ds$$

We now argue that for any bounded non-negative continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s}) d\tilde{s} ds$, exists. We show this in three steps.

(a) Given that f is continuous, it follows that the function g_1 is measurable in $\mathbb{R}_+^2$, given our assumption that $F_s(s+\tilde{s})$ is measurable, together with H_0, H_2 and f continuous (g_1 is the product of measurable functions, and thus it is itself measurable).

(b) The integral $\int_0^\infty g_1(s,\tilde{s}) d\tilde{s}$ exists given $s \in \mathbb{R}_+$. f non-negative and bounded implies that there exists a $M > 0$ such that $0 \le f \le M$. In addition, that $F_s(s+\tilde{s}) \in [0,1]$ implies $0 \le g_1(s,\tilde{s}) \le H_0(s) H_2(\tilde{s}) M \equiv \bar{g}(s,\tilde{s})$. Given $s \in \mathbb{R}_+$, the function $\bar{g}(s,\cdot)$ is integrable in $\mathbb{R}_+$, and it thus follows that $g_1(s,\cdot)$ is bounded by two integrable functions, and thus it is also integrable.

(c) From the previous step, $0 \le \int_0^\infty g_1(s,\tilde{s}) d\tilde{s} ds \le \int_0^\infty H_0(s) H_2(\tilde{s}) M d\tilde{s}$. That is, the function $g_2(s) = \int_0^\infty g_1(s,\tilde{s}) d\tilde{s}$ is bounded between 0 and $\int_0^\infty \bar{g}(s,\tilde{s}) d\tilde{s} = \hat{g}(s)$. Given that $\hat{g}(s)$ is integrable in $\mathbb{R}_+$, it provides an integrable upperbound, and it follows that the iterated integral, $\int_0^\infty \int_0^\infty g_1(s,\tilde{s}) d\tilde{s} ds$, exists.

A similar argument shows that the iterated integral in the definition of $g_0(\tau)$ exists.

Let B denote the space of continuous functions $f : \mathbb{R}_+ \to [\underline{q}, 1]$ with the sup norm. Note that this is a complete metric space. We make the following two observations about the operator T:

1) T maps B into itself.

We have already shown that for any bounded non-negative and continous f, $T\{f\}(\tau)$ exists.

Note also that $T\{f\}(\tau) \geq \underline{q} \geq 0$ and

$$T\{f\}(\tau)$$

$$\leq \underline{q} + \delta\epsilon \left[\int_\tau^\infty \int_0^\infty \int_0^{\tilde{s}} H_0(s-\tau) e^{-\epsilon \tilde{s}} dH_1(\hat{s}) d\tilde{s} ds + \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s}) d\tilde{s} ds \right]$$

$$= 1$$

where the inequality follows from using that $0 \leq f \leq 1$ and $0 \leq F_s \leq 1$. So $T\{f\} : \mathbb{R}_+ \to [\underline{q}, 1]$.

The continuity of $T\{f\}$ follows from the fact that $g_0(\tau)$ is continuous (as it is the sum a constant and the product of two continuous functions) together with the fact that $\int_\tau^\infty \int_0^\infty g_1(s, \tilde{s}) d\tilde{s} ds$ is an absolutely continuous function of τ.

2) T is a contraction mapping.

Consider two functions f and g. Then we have that

$$T\{f\}(\tau) - T\{g\}(\tau)$$

$$= \delta\epsilon \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s})(1 - F_s(s+\tilde{s}))(f(s+\tilde{s}) - g(s+\tilde{s})) d\tilde{s} ds$$

Using that $F_s(s+\tilde{s}) \in [0,1]$ we get

$$|T\{f\}(\tau) - T\{g\}(\tau)| \leq |f-g| \epsilon\delta \int_\tau^\infty \int_0^\infty H_0(s-\tau) H_2(\tilde{s}) d\tilde{s} ds$$

$$= \frac{\epsilon\delta}{(i+\lambda+\epsilon)(i+\lambda+\delta)} |f-g|$$

Thus T is a contraction mapping with modulus $\frac{\epsilon}{i+\lambda+\epsilon} \times \frac{\delta}{i+\lambda+\delta} < 1$.

It follows by the contraction mapping theorem that there exists a unique bounded and continuous function q^c such that $T\{q^c\} = q^c$ and where $q^c(\tau) \in [\underline{q}, 1]$ for all $\tau \geq 0$.

Given the existence and uniqueness of a continuous function q_c we can substitute back in the q^o equation and obtain the existence and uniqueness of q^o. It is straightforward to show that $q^o(s) \in [0,1]$ for all s.

C Continuity of q^o given construction requirement (16)

We have already shown above that q^c is continuous in any equilibrium. The continuity of q^o cannot be guaranteed in the same fashion (that is, independently of $\{F_\tau\}$). However, we can show that for any family $\{F_\tau\}$ that satisfies our construction requirement in (16), q^o must be continuous.

From the proof in Appendix B, recall that q^o can be written as:

$$q^o(s) = \epsilon \int_0^\infty \int_0^{\tilde{s}} (1 - F_s(s + \hat{s}))e^{-\epsilon\tilde{s}} dH_1(\hat{s}) d\tilde{s} + \epsilon \int_0^\infty H_2(\tilde{s}) q^c(s + \tilde{s})(1 - F_s(s + \tilde{s})) d\tilde{s}$$

where $H_1(s) = \left(1 - e^{-(i+\lambda)s}\right)$, and $H_2(s) = e^{-(i+\lambda+\epsilon)s}$.

For a family $\{F_\tau\}$ that satisfies our construction requirement in (16), the above implies that $q_o(s) = 0$ for all $s \geq T$, as $F_s(s + \hat{s}) = 1$ for all $s \geq T$ and $\hat{s} \geq 0$.

For all $s \leq T$, we have then that

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} (1 - F_s(\hat{s}))e^{-\epsilon(\tilde{s}-s)} dH_1(\hat{s} - s) d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s} - s) q^c(\tilde{s})(1 - F_s(\tilde{s})) d\tilde{s}$$

which implies that the $\lim_{s\uparrow T} q^o(s) = 0$. Thus q^o is continuous at T.

Finally, using condition (16), and letting $\hat{x}(s) = \frac{x(s)}{1-\rho(s)}$, we have that for $s < T$,

$$q^o(s) = \epsilon \int_s^T \int_s^{\tilde{s}} e^{-\int_s^{\hat{s}} \hat{x}(\tau)d\tau} e^{-\epsilon(\tilde{s}-s)} dH_1(\hat{s} - s) d\tilde{s} + \epsilon \int_s^T H_2(\tilde{s} - s) q^c(\tilde{s}) e^{-\int_s^{\tilde{s}} \hat{x}(\tau)d\tau} d\tilde{s}$$

which guarantees that q^o is a continuous function of s for $s \in [0, T)$.

Hence, we have shown that the function $q^o(s)$ associated with a family of default distributions that satisfy (16) must be continuous for all $s \geq 0$.

D Proof of Proposition 2

Proof. Consider the right-hand side of the initial value problem (IVP) defined in (10): $f(b, c) \equiv H(b, Q(b, c))$. First note that applying the chain rule delivers

$$f_b(b, c) = H_b(b, Q(b, c)) + H_q(b, Q(b, c))Q_b(b, c)$$
$$f_c(b, c) = H_q(b, Q(b, c))Q_c(b, c)$$

and thus f has continuous first derivatives in set of $(b, q) \in \mathbb{X}$ such that $H(b, q) > 0$. This implies that there exists a unique local solution $b(\tau, c^\star)$ to the IVP (10) for $c^\star > y$ and this solution is continuous and differentiable in τ and $c^\star$ in its domain of definition. See for example, Theorem 3.3 in Hale (2009).

The unique solution to (10), $b(\tau|c^\star)$, can be obtained by integration and satisfies $G(b(\tau|c^\star), c^\star) = \tau$ where the time it takes to get from $b_0 = 0$ to b (given c^*)

$$G(b, c^\star) \equiv \int_0^b \frac{1}{H(\tilde{b}, Q(\tilde{b}, c^\star))} d\tilde{b}.$$

For $c^\star \in (y, C(0, \overline{q}))$, consider the time interval of maximal existence for this solution, which in this case extends from $[0, \overline{T}]$ where $\overline{T}$ is the time where $b(\tau|c^\star)$ reaches the boundary $\overline{b}(c^\star)$ associated with a price of 1. (This time, $\overline{T}$, is finite given that H is bounded below by $c^\star - y > 0$ and $\overline{b}(c^\star)$ is finite.) Let $T(c^\star) \in [0, \overline{T})$ be such that $Q(b(T(c^\star)|c^\star), c^\star) = \overline{q}$, or $T(c^\star)$ is the amount of time it takes for Q to reach its long run value $\overline{q}$. Such a value exists as the path for b is continuous, Q is continuous and ranges from a value strictly lower than $\overline{q}$ to 1. Differentiability of Q and b together with the fact that $Q_b > 0$ and $H > 0$, implies that $T(c^\star)$ is continuous (and differentiable).

Note also that $q'(\tau|c^\star)$ as defined by equation (11) is continuous in $(\tau, c^\star)$. Using equation (12) guarantees then the corresponding $x(\tau|c^\star)$ is continuous in $(\tau, c^\star)$. As a result, the solution to the linear first order differential equation in (13) is continuous in $(\tau, c^\star)$. The above arguments then imply that $\rho(T(c^\star)|c^\star)$ is continuous in $c^\star$ for $c^\star \in (y, C(0, \overline{q}))$.

The final step for existence is to show that there are two different values c_0 and c_1 in $(y, C(0, \overline{q}))$ such that $\rho(T(c_0)|c_0) \geq 1 \geq \rho(T(c_1)|c_1)$. The existence of a $c^\star$ such that $\rho(T(c^\star)|c^\star) = 1$ follows by the established continuity of $\rho(T(c^\star)|c^\star)$.

Towards this, first note that for any $q(\tau|c) < \overline{q}$, we have that

$$x(\tau|c) > \frac{(i+\lambda)(1 - q(\tau|c))}{q(\tau|c)} > \frac{(i+\lambda)\left(\frac{\delta}{i+\lambda+\delta}\right)}{\frac{i+\lambda}{i+\lambda+\delta}} = \delta$$

where the first inequality follows from $q'(\tau|c) > 0$ and the second from $q(\tau|c) < \overline{q} = (i+\lambda)/(i+\lambda+\delta)$. It follows then that

$$\rho'(\tau|c) = (1 - \rho(\tau, c))\epsilon + \rho(\tau|c)(x(\tau|c) - \delta) > (1 - \rho(\tau|c))\epsilon$$

Thus $\rho'(\tau|c)$ is strictly positive for $\rho(\tau|c) \leq 1$. This also implies that once $\rho(\tau|c)$ has reached a value higher than 1 it cannot decrease below 1.

Now, note that for any $q(\tau|c) \leq \frac{i+\lambda}{i+\lambda+\delta+\epsilon}$, we have that

$$x(\tau|c) > \frac{(i+\lambda)(1-q(\tau|c))}{q(\tau|c)} \geq \frac{(i+\lambda)\left(\frac{\delta+\epsilon}{i+\lambda+\delta+\epsilon}\right)}{\frac{i+\lambda}{i+\lambda+\delta+\epsilon}} = \delta + \epsilon$$

where the again first inequality follows from $q'(\tau|c) > 0$ and the second from $q(\tau|c) \leq \frac{i+\lambda}{i+\lambda+\delta+\epsilon}$. It follows then that

$$\rho'(\tau|c) = (1-\rho(\tau|c))\epsilon + \rho(\tau|c)(x(\tau|c) - \delta) > \epsilon$$

for $q(\tau|c) \leq \frac{i+\lambda}{i+\lambda+\delta+\epsilon}$. Hence, we have a strictly positive lower bound to the change in $\rho(\tau|c)$ in this range.

Thus to summarize, for $q(\tau|c) \leq \frac{i+\delta}{i+\lambda+\delta+\epsilon}$, the growth of $\rho(\tau|c)$ is bounded below by a linear rate ϵ. For $q(\tau|c) \leq \overline{q}$, if $\rho(\tau|c)$ reaches a value above 1 for some τ_0, then $\rho(\tau_1|c) \geq 1$ for any subsequent $\tau_1 > \tau_0$.

From Assumption 1 part (iv), we can consider a c_0 sufficiently close to y such that $Q(0, c_0) < \frac{i+\lambda}{i+\lambda+\delta+\epsilon}$. By choosing c_0 small enough we can make $H(b(\tau|c_0), Q(b(\tau|c_0), c_0))$ arbitrary small, and thus $G(b, c_0)$ arbitrary large. The amount of time $q(\tau|c_0)$ remains below $\frac{i+\lambda}{i+\lambda+\delta+\epsilon}$, can then be made arbitrary large. Given the linear lower bound on the change in $\rho(\tau|c_0)$, it follows that for sufficiently small c_0, $\rho(\tau|c_0)$ eventually reaches above 1. Given that once it has reached one, it cannot decrease below 1, it follows that for c_0 close enough to y, $\rho(T(c_0)|c_0) > 1$.

Consider now $c_1 = C(0, \overline{q}) - \varepsilon$ for small $\varepsilon > 0$. By making ε small enough, we can make $T(c_1)$ arbitrary close to zero as $T(C(0, \overline{q})) = 0$. Now, recall from Lemma 4 that

$$Q_b = \frac{i - QH_b + \lambda(1-Q)}{H + \lambda b + QH_q} > 0$$

Assumption 1 parts (ii) and (iii) then imply that $Q_b \leq \frac{i+\lambda-QH_b}{H}$, and thus from (10)

$$q'(\tau|c_1) \leq i + \lambda + q(\tau|c_1)H_b(b(\tau|c_1), q(\tau|c_1))$$

The fact that H is Lipschitz (Assumption 1 part (i)), implies that H_b is bounded, and thus $q'(\tau|c_1) \leq M$ for some $M > 0$ independent of ε. It follows then that $x(\tau|c_1) \leq \frac{M+(i+\lambda)(1-q(\tau|c_1))}{q(\tau|c_1)} \leq \frac{M+(i+\lambda)(1-\underline{q})}{\underline{q}}$, and thus $x(\tau|c_1)$ is also bounded independent of ε. We have then that $\rho'(\tau|c_1) \leq (1-\rho(\tau|c_1))\epsilon + \rho(\tau, c_1)(M - \delta)$, and thus the growth of ρ is bounded. Hence, for ε small enough, $T(c_1)$ is close to zero, and the boundedness of $\rho'(\tau|c_1)$ implies that $\rho(\tau|c_1)$ remains below 1 for all $\tau \in [0, T(c_1)]$.

Thus, there exists c_0 and c_1 such that $\rho(T(c_0), c_0) > 1$ and $\rho(T(c_1), c_1) < 1$. Hence, there exists a $c^\star$ such that $\rho(T(c^\star), c^\star) = 1$. Note that our argument above also shows that $\rho(\tau, c^\star) < 1$ for all $\tau \in [0, T)$.

Given this value of $c^\star$, we apply Proposition 1 to show that the construction constitutes a Markov equilibrium. $\square$

E H given by (17) satisfies Assumption 1

We now show that H in equation (17) satisfies the conditions in Assumption 1 given our parameters.

For part(i): Lipschitz continuity. Consider two points $x_0 = (b_0, q_0)$ and $x_1 = (b_1, q_1)$ in $\mathbb{X}$. Let $H_0 = H(b_0, q_0)$ and $H_1 = H(b_1, q_1)$. Let $\tilde{r} = r + \lambda$ and $\tilde{i} = i + \lambda$. Let $[a]^+ = \max\{a, 0\}$, and for our parameters, $\tilde{r} > \tilde{i}$. Then,

$$\begin{aligned}
|H_0 - H_1| &= \left|[\tilde{r} - \tilde{i}/q_0]^+(y - b_0) - [\tilde{r} - \tilde{i}/q_1]^+(y - b_1)\right| \\
&= \left|\left([\tilde{r} - \tilde{i}/q_0]^+ - [\tilde{r} - \tilde{i}/q_1]^+\right)(y - b_0) + [\tilde{r} - \tilde{i}/q_1]^+(b_1 - b_0)\right| \\
&\leq \frac{\tilde{r}^2}{\tilde{i}}|q_0 - q_1| + |\tilde{r} - i| \times |b_0 - b_1| \leq \max\{\tilde{r}^2/\tilde{i}, r^\star - i\} \times (|q_0 - q_1| + |b_0 - b_1|) \\
&\leq \sqrt{2}\max\{\tilde{r}^2/\tilde{i}, r^\star - i\}|x_0 - x_1|
\end{aligned}$$

where the first inequality follows from the facts that (i) $\tilde{r}^2/\tilde{i}$ is the highest (absolute value) slope of the function $g(q) = [\tilde{r} - \tilde{i}/q]^+$ given $\tilde{r} > \tilde{i}$ and (ii) $[\tilde{r} - \tilde{i}/q]^+ \leq \tilde{r} - \tilde{i}$ as $q \leq 1$. The second inequality follows from $a + b \leq \sqrt{2}\sqrt{a^2 + b^2)}$ for $a \geq 0, b \geq 0$. Thus $M \equiv \sqrt{2}\max\{\tilde{r}^2/\tilde{i}, r^\star - i\}$ is the Lipschitz constant for all all $x_0, x_1 \in \mathbb{X}$.

Parts (ii) and (iii): These are immediate.

Parts (iv): In this case, $\underline{q} = \frac{i+\lambda}{r+\lambda}$, as $H(0, q) = 0$ for all $q \leq \underline{q}$ and $H(0, q) > 0$ for all $q > \underline{q}$. Now note that for our parameter values $\underline{q} = 0.6 < \frac{i+\lambda}{i+\lambda+\delta+\epsilon} = 0.875$.

Part (v): $H(\overline{B}, 1) = 0$ given that $\overline{B} = y$.

Part (vi): $H > 0$ requires $q \in (\underline{q}, 1]$ and $b \in [0, y)$. In this case, $H(b, q) = \left(r^\star + \lambda - \frac{i+\lambda}{q}\right)(y - b)$ which is differentiable in this domain.

F Proof of Lemma 1

Proof. Suppose $V(0) < \frac{u(y)}{r+\epsilon}$. From $b(0) = 0$, $c(0) = y + q(0)H(0, q(0)) \geq y$ from $q(0) \geq 0$ and from Assumption 1 part (iv) which guarantees $H(0, q(0)) \geq 0$. Thus, a deviation setting $F_0(0) = 1$

ensures that consumption weakly exceeds y at all dates and generates an strict improvement over $V(0)$. $\square$

G Proof of Proposition 3

Proof. The proof proceeds in four steps. We first prove three preliminary results and then the main result:

1. $V(\tau)$ is continuous.
2. $V(\tau) \geq V(0)$ for all $\tau \geq 0$.
3. For all $\tau \geq 0$, there exists $t \geq \tau$ such that $V(t) = V(0)$,
4. For all $\tau \geq 0$. $V(\tau) = V(0)$.

1) $V(\tau)$ **is continuous.**

First note that $V(\tau)$ satisfies

$$
\begin{aligned}
V(\tau) &= \sup_{T \geq 0} \int_{\tau}^{\tau+T} e^{-(\rho+\epsilon)(s-t)} u(c(s)) ds + e^{-(\rho+\epsilon)T} V(0) \\
&= \int_{\tau}^{\tau+T(\tau)} e^{-(\rho+\epsilon)(s-\tau)} u(c(s)) ds + e^{-(\rho+\epsilon)T(\tau)} V(0)
\end{aligned}
$$

for some $T(\tau) \in \mathbb{R}_+ \cup \{+\infty\}$. We then have that, for $\Delta > 0$,

$$
\begin{aligned}
V(\tau+\Delta) - V(\tau) &\geq \int_{\tau+\Delta}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} u(c(s)) ds + e^{-(\rho+\epsilon)T(\tau)} V(0) - V(\tau) \\
&= \int_{\tau+\Delta}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} u(c(s)) ds - \int_{\tau}^{\tau+T(\tau)} e^{-(\rho+\epsilon)(s-\tau)} u(c(s)) ds,
\end{aligned}
$$

where the inequality uses the weak suboptimality of $T(\tau)$ at $\tau+\Delta$. Eliminating the common terms across both integrals yields

$$
\begin{aligned}
V(\tau+\Delta) - V(\tau) &= \mathbb{1}_{\{T(\tau)<\infty\}} \int_{\tau+\max\{\Delta,T(\tau)\}}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} u(c(s))ds \\
&\qquad - \int_{\tau}^{\tau+\min\{\Delta,T(\tau)\}} e^{-(\rho+\epsilon)(s-\tau)} u(c(s))ds \\
&\geq \mathbb{1}_{\{T(\tau)<\infty\}} \left(\int_{\tau+\max\{\Delta,T(\tau)\}}^{\tau+\Delta+T(\tau)} e^{-(\rho+\epsilon)(s-(\tau+\Delta))} ds \right) \underline{u} \\
&\qquad - \left(\int_{\tau}^{\tau+\min\{\Delta,T(\tau)\}} e^{-(\rho+\epsilon)(s-\tau)} ds \right) \overline{u},
\end{aligned}
$$

where the inequality follows from the boundedness of u. Solving out the integrals, we obtain

$$
\begin{aligned}
V(\tau+\Delta) - V(\tau) &= \mathbb{1}_{\{T(\tau)<\infty\}} \left(\frac{e^{(\rho+\epsilon)(\Delta-\max\{\Delta,T(\tau)\})} - e^{-(\rho+\epsilon)T(\tau)}}{\rho+\epsilon} \right) \underline{u} \\
&\qquad - \left(\frac{1-e^{-(\rho+\epsilon)\min\{\Delta,T(\tau)\}}}{\rho+\epsilon} \right) \overline{u} \\
&\geq - \sup_{T\geq 0} \left| \frac{e^{(\rho+\epsilon)(\Delta-\max\{\Delta,T\})} - e^{-(\rho+\epsilon)T}}{\rho+\epsilon} \right| |\underline{u}| - \sup_{T\geq 0} \left| \frac{1-e^{-(\rho+\epsilon)\min\{\Delta,T\}}}{\rho+\epsilon} \right| |\overline{u}| \\
&= - \left| \frac{1-e^{-(\rho+\epsilon)\Delta}}{\rho+\epsilon} \right| (|\overline{u}| + |\underline{u}|) > -\infty.
\end{aligned}
$$

A similar argument provides the same lower bound for $V(\tau) - V(\tau+\Delta)$, and thus

$$
|V(\tau+\Delta) - V(\tau)| \leq \left| \frac{1-e^{-(\rho+\epsilon)\Delta}}{\rho+\epsilon} \right| (|\overline{u}| + |\underline{u}|).
$$

Given that the right-hand side is continuous and goes to zero as $\Delta \to 0$, this guarantees the continuity of $V(\tau)$.

2) $V(\tau) \geq V(0)$ for all $\tau \geq 0$.

Choosing $F_\tau(\tau) = 1$ guarantees a value equal to $V(0)$.

3) For all $\tau \geq 0$, there exists $t \geq \tau$ such that $V(t) = V(0)$.

First, we show there exists t such that $F_\tau(t) > 0$. Toward a contradiction, suppose $F_\tau(t) = 0$ for all $t \geq \tau$ for some $\tau \geq 0$. This implies for all $t \geq \tau$ that $q(t) = 1$.

Lemma 3 shows that $b(\tau)$ converges to some b_{ss}. Given that $q(t) = 1 > \underline{q}$, it follows from Assumption 1(iv) that $b_{ss} > 0$. Thus for $t > \tau$ large enough $c(t) \to y - ib_{ss} < y$. Hence, there exists $s > \tau$ such that $c(t) < y$ for all $t > s$. This implies that $V(s) < u(y)/(r+\epsilon)$, a contradiction of Lemma 1. And thus, there exists t such that $F_\tau(t) > 0$.

Next, define $s \equiv \inf\{t \geq \tau | F_\tau(t) > 0\}$. The previous result guarantees that such an s is finite. Suppose that $V(s) > V(0)$. The continuity of V implies there exists $\Delta > 0$ such that $V(t) > V(0)$ for all $t \in [s, s+\Delta]$. Optimization then implies $F_s(s+\Delta) = 0$. From the definition of s, $F_\tau(s^-) = 0$. Therefore, since $(1 - F_\tau(s+\Delta)) = (1 - F_\tau(s^-))(1 - F_s(s+\Delta))$, $F_\tau(s+\Delta) = 0$. This contradicts the definition of s. And thus, $V(s) = V(0)$.

4) For all $\tau \geq 0$, $V(\tau) = V(0)$.

Steps 1) through 3) establish that if $V(\tau) > V(0)$ for any $\tau > 0$, there must be an open interval containing τ, $(s, s+\Delta)$, such that $V(s) = V(s+\Delta) = V(0)$ and $V(t) > V(0)$ for all $t \in (s, s+\Delta)$. (That is, the function V must have a continuous "hill" containing τ.) Since $V(t) > V(0)$ for all $t \in (s, s+\Delta)$, optimization implies $F_m(n) = 0$ for all $(m, n) \in (s, s+\Delta)^2$ with $n \geq m$. This implies (for $t \in (s, s+\Delta)$) that

$$q(t) = \int_t^{s+\Delta} e^{-(i+\lambda)(m-t)}(i+\lambda)dm + e^{-(i+\lambda)(s+\Delta-t)}q(s+\Delta) = 1 - e^{-(i+\lambda)(s+\Delta-t)}(1 - q(s+\Delta)).$$

Since $q(s+\Delta) < 1$ (from the positive probability of default at some time shown in Step 3), $q(t)$ is a continuous strictly decreasing function on $(s, s+\Delta)$.

Next we show that $b(\tau)$ is weakly increasing on $(s, s+\Delta)$. We proceed according to the following steps.

(i) First, we show that if for some $m \in (s, s+\Delta)$, $b'(m) < 0$, then $b'(n) \leq 0$ for all $n \in (m, s+\Delta)$.

  Towards a contradiction here, let there be such an n for some m. Let $\hat{n}$ be such that $\hat{n} = \min\{t | b'(t) = 0, m < t \leq n\}$. Note that $\hat{n}$ exists and $\hat{n} \in (m, n]$, by the existence of $b'(n) \geq 0$ and $b'(m) < 0$ and the continuity of H (Assumption 1 part (i)) together with both q and b continuous. This implies that $b'(t) < 0$ for all $t \in (m, \hat{n})$, and thus $b(\hat{n}) < b(m)$, from the integral form (9). Given that q is strictly decreasing on this interval, it follows that $H(b(\hat{n}), q(\hat{n})) \leq H(b(m), q(m))$ by Assumption 1 parts (ii) and (iii). And thus $H(b(\hat{n}), q(\hat{n})) < 0$, violating that $b'(\hat{n}) = 0$.

(ii) The above implies that if $b'(m) < 0$ for $m \in (s, s+\Delta)$, then $b'(n) < 0$ for $n \in (m, s+\Delta)$, and thus debt is decreasing over the rest of the interval. That $H(b(t), q(t)) \leq 0$ implies in addition that $c(t) = C(b(t), q(t)) \leq y - ib(t) - \lambda(1 - q(t))b(t)$ over the interval. Using that $q(t) < 1$ and that $b(t) \geq 0$ (by our equilibrium restriction), we have then that $c(t) < y$ for $t \in [m, s+\Delta)$.

But note for all $m \in (s, s+\Delta)$

$$V(m) = \int_m^{s+\Delta} e^{-(r+\epsilon)(t-m)}[u(c(t)) - (r+\epsilon)V(0)]dt + V(0). \tag{22}$$

If $c(t) < y$ for all $t \in [m, s+\Delta)$, then $[u(c(t)) - (r+\epsilon)V(0)] < 0$ for all such t. This contradicts $V(m) > V(0)$.

Thus, $b'(t) \geq 0$ for all $t \in (s, s+\Delta)$, or $b(\tau)$ is weakly increasing on the open interval.

(iii) Now we argue that $c(\tau)$ is weakly decreasing on $(s, s+\Delta)$. To see this, for $t_0 < t_1$ in the interval, we have

$$\begin{aligned} c(t_0) &= y - ib(t_0) + q(t_0)H(b(t_0), q(t_0)) - \lambda(1-q(t_0))b(t_0) \\ &\geq y - ib(t_1) + q(t_1)H(b(t_0), q(t_0)) - \lambda(1-q(t_0))b(t_1) \\ &\geq y - ib(t_1) + q(t_1)H(b(t_0), q(t_0)) - \lambda(1-q(t_1))b(t_1) \end{aligned}$$

where we used that $b(\tau)$ is weakly increasing and $q(\tau)$ is strictly decreasing in the interval together with $H \geq 0$ (debt is weakly increasing), $b \geq 0$ (equilibrium condition), and that $q \leq 1$. Then Assumption 1 part (ii) and (iii) imply that $H(b(t_0), q(t_0)) \geq H(b(t_1), q(t_1))$, and thus $c(t_0) \geq c(t_1)$.

(iv) Finally, we know

$$V(s) = V(0) = \int_s^{s+\Delta} e^{-(r+\epsilon)(t-s)}[u(c(t)) - (r+\epsilon)V(0)]dt + V(0), \tag{23}$$

which implies $\int_s^{s+\Delta} e^{-(r+\epsilon)(t-s)}[u(c(t)) - (r+\epsilon)V(0)]dt = 0$.

That $V(t) > V(0)$ for any $t \in (s, s+\delta)$, implies that $\int_t^{s+\Delta} e^{-(r+\epsilon)(t-t)}[u(c(t)) - (r+\epsilon)V(0)]dt > 0$ and $\int_s^t e^{-(r+\epsilon)(t-t)}[u(c(t)) - (r+\epsilon)V(0)]dt < 0$, as their sum is zero. But these mean that there is a $t_0 \in (s, t)$ such that $u(c(t_0)) < (r+\epsilon)V(0)$ and there is a $t_1 \in (t, s+\Delta)$ such that $u(c(t_1)) > (r+\epsilon)V(0)$.

Step (iv) contradicts step (iii) that requires $c(\tau)$ weakly decreasing on $(s, s+\Delta)$. Thus, there cannot be a $\tau > 0$ such that $V(\tau) > V(0)$. □

H Proof of Lemma 2

Proof. Consider such a τ and Δ. The fact that the opportunistic type is indifferent between defaulting or not in $[\tau, \Delta]$ implies that

$$\int_{\tau}^{\tau+\Delta'} [u(c(t)) - (r+\epsilon)V]\, dt = 0$$

for any $\Delta' \in (\tau, \Delta]$, which implies that $u(c(t)) = (r+\epsilon)V$ a.e., implying the result. $\square$

I Proof of Proposition 4

Proof. We begin by constructing an increasing sequence of dates $\{\tau_1, \tau_1+\Delta_1, \tau_2, \tau_2+\Delta_2, \ldots\}$ where $F_s(s) < 1$ between dates τ_i and $\tau_i + \Delta_i$ and $F_s(s) = 1$ between dates $\tau_i + \Delta_i$ and τ_{i+1}. This sequence may contain an infinite or finite number of elements.

To this end, let $\tau_1 \equiv \inf\{\tau \geq T | F_\tau(\tau) < 1\}$. (If the set defining τ_1 is empty, the result is proved.) Next, define $\Delta_1(s) = \inf\{\hat{s} \geq 0 | F_{\tau_1+s}(\tau_1 + s + \hat{s}) = 1)$ and $\Delta_1 = \inf_{s>0} \Delta_1(s)$. If Δ_1 does not exist, the sequence is $\{\tau_1\}$. If Δ_1 exists, that $F_{\tau_1}(s)$ is right-continuous ensures $\Delta_1 > 0$. Continuing, let $\tau_2 \equiv \inf\{\tau \geq \tau_1 + \Delta_1 | F_\tau(\tau) < 1\}$, where if τ_2 does not exist, the sequence is $\{\tau_1, \tau_1 + \Delta_1\}$. Likewise, $\Delta_2(s) = \inf\{\hat{s} \geq 0 | F_{\tau_2+s}(\tau_2 + s + \hat{s}) = 1)$ and $\Delta_2 = \inf_{s>0} \Delta_2(s)$, and so on, where at each step if τ_i or Δ_i does not exist, the sequence ends. From the definition of τ_i, either $\rho(\tau_i^-) = 1$ or $\rho(\tau_i) = 1$. If $\rho(\tau_i^-) = 1$, $q(\tau_i) = q^c(\tau_i)$ from (5). If $\rho(\tau_i) = 1$, $q(\tau_i^+) = q^c(\tau_i^+) = q^c(\tau_i)$ from the continuity of q^c. Likewise, from the definition of Δ_i, $\rho(\tau_i + \Delta_i) = 1$ and thus $q(\tau_i + \Delta_i) = q^c(\tau_i + \Delta_i)$.

First suppose, Δ_i does not exist for some τ_i. This implies after τ_i, $F_\tau(\tau) < 1$. Lemma 2 then implies $c = c^\star > y$ for all $\tau > \tau_i$ which violates the debt limit.

Next suppose the sequence is of finite length $\{\tau_1, \tau_1 + \Delta_1, \ldots, \tau_n, \tau_n + \Delta_n\}$. Here, $q(\tau_n + \Delta_n) = q^c(\tau_n + \Delta_n) = \frac{i+\lambda}{i+\lambda+\delta}$, and $q(\tau_n) = q^c(\tau_n) = \frac{i+\lambda}{i+\lambda+\delta} + \int_{\tau_n}^{\tau_n+\Delta_n} e^{-(i+\lambda+\delta)(s-\tau_n)} \delta q^o(\tau_n + s) ds$, which implies $q(\tau_n) \geq q(\tau_n + \Delta_n)$. This contradicts $c(\tau) = c^\star > y$ for τ between τ_n and $\tau_n + \Delta_n$ (since if debt is rising, bond prices must be increasing for consumption to stay constant given Assumption 1(ii) and (iii)).

Next, suppose $\sum_{i=1}^{\infty}(\tau_{i+1} - (\tau_i + \Delta_i)) = \infty$, or there is an infinite amount of time after T where $F_\tau(\tau) = 1$, and let

$$\tilde{q}(\tau) \equiv q(\tau) - \frac{i+\lambda}{i+\lambda+\delta}.$$

Using that $q^o(\tau) = 0$ for $\tau \in (\tau_i + \Delta_i, \tau_{i+1})$, we have that $\tilde{q}(\tau_{i+1}) = e^{(i+\lambda+\delta)(\tau_{i+1}-(\tau_i+\Delta_i))} \tilde{q}(\tau_i + \Delta_i)$ from the definition of $\tilde{q}$ and the integral form of q^c. That is, between dates $\tau_i + \Delta_i$ and τ_{i+1}, $\tilde{q}$ grows at the rate $i + \lambda + \delta$. Further, between dates τ_i and $\tau_i + \Delta_i$, q is strictly increasing from $c(\tau) = c^\star > y$.

Thus $\tilde{q}$ is unbounded, contradicting $q(\tau) \leq 1$.

Thus finally suppose $\sum_{i=1}^{\infty} \Delta_i = \infty$ with $\sum_{i=1}^{\infty}(\tau_{i+1} - (\tau_i + \Delta_i)) < \infty$, or there is an infinite amount of time after T where where $F_\tau(\tau) < 1$ and a finite amount of time when $F_\tau(\tau) = 1$. Here, $c(\tau) = c^\star > y$ and thus debt $b(\tau)$ is positive and grows at rate bounded by i for an infinite amount of time, and $b(\tau)$ falls at most at the rate λ for a finite amount of time. Taken together, these imply that the debt limit is violated. □

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