Computing sovereign debt models: Why so hard?

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VERY PRELIMINARY AND INCOMPLETE

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Abstract

Sovereign debt models with long-duration bonds are notoriously hard to compute. Using a simplified environment of the standard Eaton and Gersovitz (1981) model with outside option shocks, we show that equilibria in pure strategy may not exist, explaining the lack-of-convergence issues encountered in the quantitative literature. We propose an algorithm for computing mixed-strategy equilibria. For some parameterizations, we uncover millions.

1 Introduction

2 Debt and Default with Long Duration Bonds

We begin by stating the standard sovereign debt model, based on the original contribution by Eaton and Gersovitz (1981) and subsequently developed by Arellano (2008) and Aguiar and Gopinath (2006). This initial version of the model is based on incomplete markets where the government of a small open economy could borrow externally using a defaultable but otherwise uncontingent real bond. The bond was assumed to be of one-period maturity. Subsequent contributions (Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012), among others) relaxed the assumption of one-period bonds, allowing for the introduction of higher maturities. As highlighted in Chatterjee and Eyigungor (2012), longer maturity allows the model to much better match the level of debt, the behavior of consumption over the cycle and the interest spread between the small open economy's bond and a risk-free bond.

The basic model is as follows. A small open economy receives an endowment of $y(s)$ every period, where $s \in \mathbb{R}^N$. We assume that s follows a first order Markov process, and let $F(s'|s)$ be the associated cumulative probability distribution of the next period state s' conditional on today's state s. The government of the small open economy decides whether to repay the inherited debt or not. In case of repayment, the government auctions bonds at a price q, and uses the revenue generated together with the economy's endowment to consume and pay the maturing debt. The debt consists of unit of exponential bonds that decay at rate δ. We denote debt by b and restrict attention to $b \leq \overline{B}$, where $\overline{B}$ is set to a sufficiently large number such as to rule out Ponzi schemes.

We focus on Markov equilibria, where the strategy of the government and the price of the outstanding bonds are functions only of payoff relevant states. In our case, b and s. The strategy of the government consists of a default probability $D(b, s)$ as well as a cumulative probability distribution $\Pi(b'|b, s)$ detailing the probability that, conditional on no default, the government leaves a given level of debt below b' for the next period as a function of the current state (b, s). As usual, the price of the bonds is then a function $q(b', s)$ of the current s and the debt level b' chosen for the following period.

We denote by $v^R(b, s)$ the government's value function associated with repayment of the debt, b, in state s; and denote by $v^D(s)$ the government's value associated with default. The literature usually endogenizes the value of v^D by postulating a problem where the government, once it defaults it is shut out from financial markets, suffers output costs and faces a potential re-entry probability to financial markets. In what follows, we simplify the analysis and assume that $v^D(s)$ is given.

The budget constraint in state (b, s) conditional on repayment is given by the following consumption level as a function of the state s, the inherited debt b, and the new debt level b':

$$c(b', b, s) \equiv y(s) - (r + \delta)b + q(b', s)(b' - (1 - \delta)b)$$

The government's value of repayment, V^R, satisfies the following Bellman equation:

$$v^R(b, s) = \max_{c, b' \in [0, \overline{B}]} \left\{ u(c(b', b, s)) + \beta \int_{s'} \max\{v^R(b', s'), v^D(s')\} dF(s'|s) \right\} \quad (1)$$

The equilibrium price satisfies:

$$q(b, s) = \int_{b'} \int_{s'} (1 - D(b, s')) \frac{r + \delta + (1 - \delta) q(b', s')}{R} dF(s'|s) d\Pi(b'|b, s) \quad (2)$$

Government's optimality requires that the government chooses the repayment probability optimally. That is:

$$D(b,s') \in \begin{cases} \{0\} & \text{if } v^R(b,s) > v^D(s) \\ [0,1] & \text{if } v^R(b,s) = v^D(s) \\ \{1\} & \text{otherwise} \end{cases} \tag{3}$$

In addition, any debt level b' in the support of $\Pi(.|b,s)$ (that is a debt level that is chosen with positive probability/density) must attain the maximum in the government's problem:

$$u(c(b',b,s)) + \beta \int_{s'} \max\{v^R(b',s'), v^D(s')\} dF(s'|s) = v^R(b,s) \tag{4}$$

for all b' in the support of $\Pi(.|b,s)$.

With the above, we can define a Markov equilibrium:

Definition 1. A *Markov equilibrium* is a repayment value function, v^R, a default probability function, D, a probability distribution over debt choices, Π, and a price function, q, such that (1), (2), (3) and (4) hold.

We next show a result that will prove helpful in later sections: To characterize an equilibrium, it is without loss of generality to restrict attention to probability distribution functions $\Pi(b'|b,s)$ that put strictly positive mass on at most two points in $[0,\overline{B}]$:

Proposition 1 (Two suffices)**.** *Consider a Markov equilibrium* (v^R, D, Π)*, then there exists a Markov equilibrium* $(v^R, D, \hat{\Pi})$ *where for all* (b,s)*,* $\hat{\Pi}(b'|b,s)$ *puts positive mass on at most two* b'*.*

Proof. TO BE COMPLETED. □

Chatterjee and Eyigungor (2012) argue that there exists a Markov equilibria when mixed strategies are allowed (see footnote 13 of that paper). However, they say that "While this approach [that is, allowing for mixed strategies] solves the existence issue, it does not appear to be computationally tractable. In particular, computing mixed strategies when the 'support points' of the mixed strategy are not known in advance—and the choice set is very large—seems to be a challenging task." Our Proposition 1 alleviates the computational problem by reducing the number of points in the support, but the problem remains still intractable computationally.

For this reason, the literature usually restrict attention to pure strategies Markov equilibria when computing equilibria numerically. Pure strategy Markov equilibria basically require that the probability distributions $\Pi(.|b,s)$ be degenerate – that is, only one level of debt is chosen with positive probability in any state. In addition, the literature sometimes also restricts attention to equilibria where the government repays if indifferent; that is, $D(b,s')=0$ if $v^R(b,s)=v^D(s)$. A typical approach is to define a Pure strategy equilibrium as a fixed point. Given an initial guess of v^D and a q functions, one could use the value function (1) and the price equation (2) to iterate. However, convergence under this approach is not guaranteed, as the operator implicitely defined by this fixed point problem can be shown not to be a contraction (or even monotone). More problematic, convergence may not occur because no Pure strategy Markov equilibria exists, a fact suspected by Chatterjee and Eyigungor (2012) (See text around their footnote 12). To overcome the lack of convergence in numerical methods, Chatterjee and Eyigungor (2012) and CITE SANCHEZ propose modifications to the environment that overcome the issues.

In what follows, we however pursue a different approach. We work with a simplified version of this environment and show first that equilibria generically do not exist for a range of parameter values when restricted to pure strategies. This suggests that the convergence problems faced by the literature may be due to lack of existence of the object that is been computed, as suspected by Chatterjee and Eyigungor (2012). And second, we show that in our simplified environment, once we allowed for mixed strategies, we can compute equilibria directly and there are many.

3 An Environment with just Outside Option Shocks

In what follows, we proceed to simplify the environment. We impose the following assumptions:

Assumption 1. *The following holds:*

(i) Endowment is constant, $y(s)=y$ *for all* $s\in S$.

(ii) Exogenous state is i.i.d., $F(s'|s)=F(s')$ *for all* $s,s'\in S$.

Note that with this assumption, the only effect of the shock s is to potentially change the outside option value $v^D(s)$. An environment like this, but with a portfolio of maturities available to the government (including a one-period bond), was analyzed in detail in Aguiar et al. (2018). The ability to issue one-period debt was important in the tractability of that paper and proved useful in characterizing equilibria. In this environment, however, we are focused in the case where the government does not have access to a one-period bond, a common restriction in the applied literature.

One may think that in this simpler environment, showing the existence of an equilibrium in pure strategies and numerically computing equilibria should be a straight-forward matter. As we show next, this environment suffers from the same computational issues as the more general model.

Using Assumption 1 allows to drop the current state s as an argument for the value function, v^R, the price function, q, and the policy function Π. The reason for the latter is that, conditional on repayment, the inherited state is payoff irrelevant, as it only determines the current outside option which does not affect the repayment problem. Note that this implies that $c(b', b, s)$ can be now written as:

$$c(b', b) \equiv y - (r + \delta)b + q(b')(b' - (1 - \delta)b)$$

With this, we can write the government problem as follows:

$$v^R(b) = \max_{b' \leq \overline{B}} \left\{ u(c(b', b)) + \beta \int_{s'} \max\{v^R(b'), v^D(s')\} dF(s') \right\}$$

Equilibrium prices now solve:

$$q(b) = \int_{b'} \int_{s'} (1 - D(b, s')) \frac{r + \delta + (1 - \delta)q(b')}{R} dF(s') d\Pi(b'|b)$$

And where

$$u(c(b', b)) + \beta \int_{s'} \max\{v^R(b'), v^D(s')\} dF(s') = v^R(b)$$

for all b' in the support of $\Pi(.|b)$.

We can then argue that $v^R(b)$ is strictly decreasing in b:

Lemma 1. *Suppose Assumption 1 holds. In a Markov equilibrium, $v^R(b)$ is strictly decreasing in b.*

Proof. TO BE COMPLETED. This can be extended to continuity. We just need consumption to be strictly positive. □

3.1 A Dual Representation

We now proceed to show a dual representation of equilibria that we will exploit in our analysis.[1] Let us denote by $B(v)$ the inverse of the value function $v^R(b)$, which exists given

[1] This is similar to the dual representation used in Aguiar et al. (2018) and Aguiar and Amador (2019), but in this case, using a long term bond.

Lemma 1.

Let us define $\tilde{c}(v, v')$ as follows:

$$c(v', v) \equiv u^{-1}\left(v - \beta \int_{v^D} \max\{v', v^D\} d\tilde{F}(v^D)\right) \tag{5}$$

Then, we can write the dual problem of the government as follows:

$$B(v) = \max_{v'} \left\{ \frac{y - c(v', v) + \tilde{q}(v')B(v')}{r + \delta + \tilde{q}(v')(1 - \delta)} \right\} \tag{6}$$

where $\tilde{F}$ represents the CDF over outside option values.

The equilibrium price function must satisfy $\tilde{q}(v) = q(B(v))$. Using this, we can rewrite the price equation as:

$$\tilde{q}(v) = \left[\int_{v^D} (1 - \tilde{D}(v, v^D)) d\tilde{F}(v^D)\right] \times \left[\int_{v'} \frac{r + \delta + (1 - \delta)\tilde{q}(v')}{R} d\tilde{\Pi}(v'|v)\right] \tag{7}$$

The default probability is as before:

$$\tilde{D}(v, v^D) \in \begin{cases} \{0\} & \text{if } v > v^D \\ [0, 1] & \text{if } v = v^D \\ \{1\} & \text{otherwise} \end{cases} \tag{8}$$

Given that any v' used in an equilibrium with positive probability must be consistent with the same level of debt $B(v)$, it follows that the probability function $\tilde{\Pi}$ must be such that

$$\frac{y - c(v', v) + \tilde{q}(v')B(v')}{r + \delta + \tilde{q}(v')(1 - \delta)} = B(v) \tag{9}$$

for all v' in the support of $\tilde{\Pi}(v'|v)$.

We conclude this section with summarizing the above:

Lemma 2. *Suppose Assumption 1 holds. Consider a Markov equilibrium* $\{v^R, q, D, \Pi\}$*, then, there exists functions* B*,* $\tilde{q}$*,* $\tilde{D}$*, and* $\tilde{\Pi}$ *that satisfy* (6)*,* (7)*,* (8) *and* (9) *and such that* $B(v^R(b)) = b$ *and* $\tilde{q}(v^R(b)) = q(b)$ *for all* $b \leq \overline{B}$.

3.2 Non-Existence of Pure Strategy Markov Equilibria

Let us consider the dual problem, when restricted to pure strategies. With pure strategies, there is a policy function $g(v)$ that for any current value v for the government it returns its associated value next period v', conditional on no default.

The government's problem remains the same as in (6). However we have that $g(v)$ must satisfy:

$$\frac{y - c(g(v), v) + \tilde{q}(g(v))B(g(v))}{r + \delta + (1-\delta)\tilde{q}(g(v))} = B(v)$$

And the prices are such that

$$\tilde{q}(v) = \left[\int_{v^D} (1 - \tilde{D}(v, v^D))d\tilde{F}(v^D)\right] \times \frac{r + \delta + (1-\delta)\tilde{q}(g(v))}{R}$$

Note that if $\tilde{F}$ has continuous density, the term in square brackets becomes $\tilde{F}(v)$; as the indifference points have zero mass.

[DISCUSS NON EXISTENCE OF PURE STRATEGIES]

What would happen if a researcher attempts to numerically compute a pure strategy equilibrium in this environment using a standard iterative procedure, when the equilibrium does not exist? In this case, we obtain a lack of convergence that is very similar in nature to the problem that arises when computing a more general model where the endowment is stochastic.

Figure 1 shows the result from an iteration of the value and price functions (details of the comptuations are provided in the Appendix).

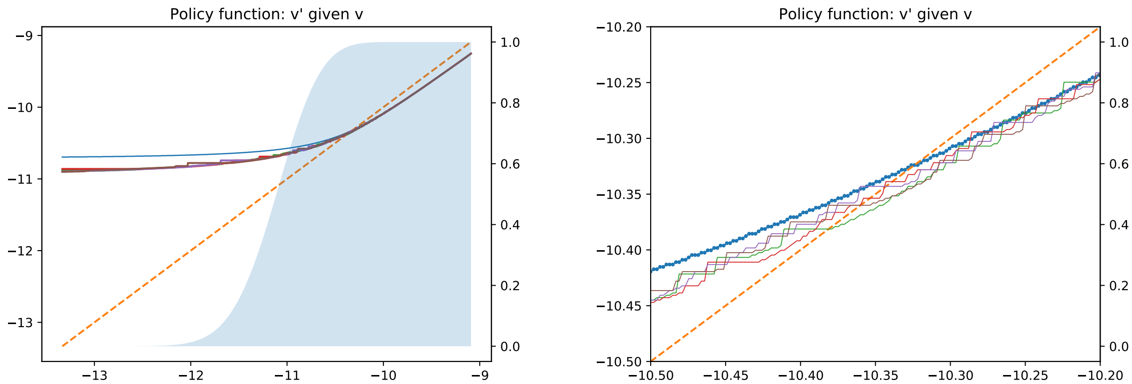
3.3 Computing Mixed Strategy Equilibria: Too Many

Although mixed strategy Markov equilibria exists, they are hard to compute numerically. As stated early on, it is difficult to know where the support of the policy distribution $\hat{\Pi}$ is.

In this section we are able to overcome this problem. We first conjecture that mixing at a point v always involve the stationary policy value $v' = v$. That is, in an equilibrium, the government only mixes between a policy away from the 45 degree line, and the policy at the 45 degree line. With this guess, we construct an algorithm that is able to recover the equilibria. The details of the algorithm are in the Appendix.

Using the same parameterization as in Figure 1, we recover 325,913 equilibria when restricting the grid to 2000 points. Higher number of grids points quickly sends the number of equilibria above millions. Figure 2 plots one of the mixed strategy equilibria.

Figure 1: Lack of Convergence with Pure Strategies



(a) Policy function

(b) Zoomed around the 45 degree line.

This figure shows the cycles generated by iterating on the value and price functions using the functional equations. The parameters are $u(c) = -1/c$, $\beta = 0.9$, $R = 1.05$, $y = 1$, $\delta = 0.6$, $v^D = u(\tau y)/(1-\beta)$ where τ is a Truncated Normal with mean $\mu = 0.9$, $\sigma = 0.03$ and truncated within $[0.75, 0.999]$. We use a grid of 2,000 points for v in $[u(0.75 * y)/(1-\beta), u(1.1 * 0.999 * y)/(1-\beta)]$. In this case, the numerical procedure cycles every 7 iterations and does not converge. The blue solid line represents the efficient outcome (which obtains when $\delta = 1$). The blue shade in panel (a) represents the probability of repayment $\tilde{F}(v)$.

4 Continuous Time

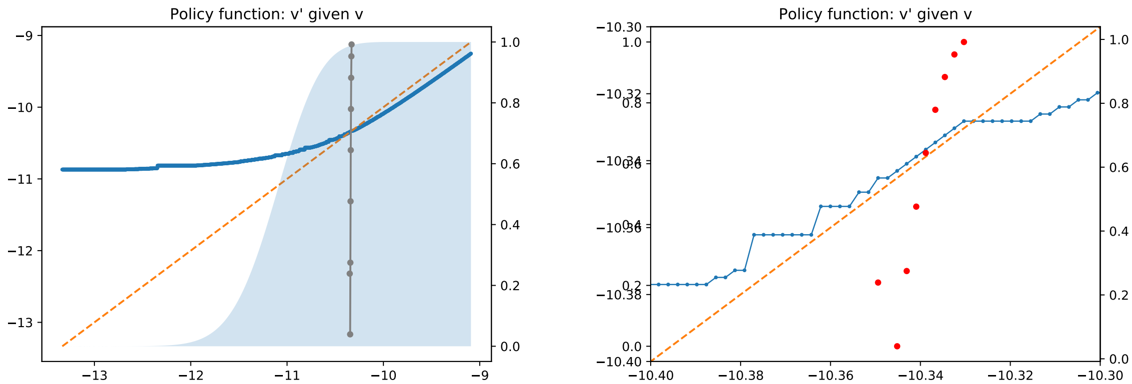
We conjecture that the lack of existence and the high number of mixed strategy equilibria are artifacts of the discreteness of time. ...

5 Conclusion

In a simplified version of the standard Eaton and Gersovitz (1981) sovereign debt model, we show that

- Pure Strategy Equilibria may not exist.
- Mixed Strategy Equilibria are many and can be computed.

Figure 2: A Mixed Strategy Equilibrium



(a) Policy function

(b) Zoomed around the 45 degree line.

This figure shows a mixed strategy equilibrium. The parameters are $u(c) = -1/c$, $\beta = 0.9$, $R = 1.05$, $y = 1$, $\delta = 0.6$, $v^D = u(\tau y)/(1-\beta)$ where τ is a Truncated Normal with mean $\mu = 0.9$, $\sigma = 0.03$ and truncated within $[0.75, 0.999]$. We use a grid of 2,000 points for v in $[u(0.75 * y)/(1-\beta), u(1.1 * 0.999 * y)/(1-\beta)]$. The blue shade in panel (a) represents the probability of repayment $\tilde{F}(v)$. The dots represent the probability of mixing and staying at the same value.

A Proofs

B Numerical Algorithms

The algorithm works as follows:

1. Construct all local savings equilibria.
2. Construct all local pure strategy borrowing equilibria.
3. Construct all local mixed strategy borrowing equilibria.
4. Find all possible connections between each pair of local equilibria.
5. Use full lists of local equilibria and possible connections between them to construct the complete set of full equilibria.

In the above, a "local equilibrium" is a set of equilibrium objects $(B(v), q(v), \Pi(v'|v), g(v))$ which satisfy the functional equations and optimality conditions describing the program on a restricted domain $[v_L, v_H]$. A "local savings equilibrium" is a local equilibrium which has $\Pi(v_H|v_H) = 1$, and a "local borrowing equilibrium" is a local equilibrium which has

$\Pi(v_L|v_L) = 1$. Note that, since policies are monotone, every full equilibrium can be broken down into some number of local borrowing and savings equilibria. A possible connection between a pair of equilibria is for some two equilibria satisfying $\{v_L, ...v_H\} \cup \{v'_L, ..., v'_H\} = \{min\{v_L, v'_L\}, ..., max\{v_H, v'_H\}$ (i.e. their domains form a contiguous region of the state space) a point v such that if the system of objects $(B(v), q(v), \Pi(v'|v), g(v))$ formed by combining the two around v is also a local equilibrium. By "combining the two around v", we mean assigning to v_k values strictly below v the values associated with one equilibrium and to v_k values weakly greater than v the values associated with the other.

To construct local savings equilibria, we use the following strategy to obtain a complete representation of $[v_L, v_H]$ pairs and the associated equilibrium objects. For each $k \in \{2, 3, ..., N_v\}$:

1. Suppose that $v_H = v_k$.
2. Calculate the stationary value and price at v_k, $B^{st}(v_k)$ and $q^{st}(v_k)$, respectively:

$$q^{st}(v_k) = \frac{F(v_k)(r+\delta)}{R-(1-\delta)F(v_k)} \tag{10}$$

$$B^{st}(v_k) = \frac{R - F(v_k) + \delta F(v_k)}{r+\delta} \frac{y - c(v_k, v_k)}{R - F(v_k)} \tag{11}$$

3. Set $g(v_k) = v_k, B(v_k) = B^{st}(v_k), q(v_k) = q^{st}(v_k), \Pi(.|v_k) = 1$ and store them.
4. Set $j = k - 1$ and $v_H(k) = v_k$
5. Calculate the stationary value and price at v_j, $B^{st}(v_j)$ and $q^{st}(v_j)$.
6. Solve the dual problem assuming that $g(v_j) > v_j$.
7. Calculate the price at $q_{pure}(v_j)$ implied by the above solution. Note that we always have $q_{pure}(v_j) > q^{st}(v_j)$.
8. Calculate the deviation value $\hat{B}(v_j, q_{pure}(v_j))$ as:

$$\hat{B}(v_j, q) = \frac{y - c(v_j, v_j) + qB(v_j)}{r + \delta + (1-\delta)q} \tag{12}$$

9. If $\hat{B}(v_j) > B(v_j)$ and $B^{st}(v_j) > B(v_j)$, set $v_L(k) = v_{l+1}$, move to the next k and return to step 1.
10. If $\hat{B}(v_j) \leq B(v_j)$, set $\Pi(g(v_j)|v_j) = 1$ and $q(v_j) = q_{pure}(v_j)$. Then go to the last step in this list.

11. If $\hat{B}(v_j) > B(v_j)$ and $B^{st}(v_j) < B(v_j)$, then there exists a $q_{mix} \in [q_{pure}(v_j), q^{st}(v_j)]$ such that $\hat{B}(v_j, q) = B(v_j)$ and a probability of choosing $g(v_j)$ $\pi \in [0,1]$ such that $q(v_j) = q_{mix}$. Calculate q_{mix}, π.

12. Set $q(v_j) = q_{mix}$, $\Pi(g(v_j)|v_j) = \pi$, and $\Pi(v_j|v_j) = 1 - \pi$.

13. Store $g(v_j), B(v_j), q(v_j), \Pi(.|v_j)$. Then, if $j = 1$, set $v_L(k) = v_1$, move to the next k and return to step 1. Otherwise, deincrement j (i.e. set $j = j - 1$) and return to step 5.

To construct local pure strategy borrowing equilibria, we use a reduced version of the above. In particular, for every $k \in \{1, 2, ..., N_v - 1\}$:

1. Suppose that $v_L = v_k$.

2. Calculate the stationary value and price at v_k, $B^{st}(v_k)$ and $q^{st}(v_k)$, respectively:

$$q^{st}(v_k) = \frac{F(v_k)(r+\delta)}{R - (1-\delta)F(v_k)} \tag{13}$$

$$B^{st}(v_k) = \frac{R - F(v_k) + \delta F(v_k)}{r+\delta} \frac{y - c(v_k, v_k)}{R - F(v_k)} \tag{14}$$

3. Set $g(v_k) = v_k, B(v_k) = B^{st}(v_k), q(v_k) = q^{st}(v_k), \Pi(.|v_k) = 1$ and store them.

4. Set $j = k + 1$ and $v_L(k) = v_k$

5. Solve the dual problem assuming that $g(v_j) > v_j$.

6. Calculate the price at $q_{pure}(v_j)$ implied by the above solution.

7. Calculate the deviation value $\hat{B}(v_j, q_{pure}(v_j))$ as before:

8. If $\hat{B}(v_j) > B(v_j)$, set $v_H(k) = v_{j-1}$, move to the next k and return to step 1.

9. If $\hat{B}(v_j) \leq B(v_j)$, set $\Pi(g(v_j)|v_j) = 1$, $q(v_j) = q_{pure}(v_j)$.

10. Store $g(v_j), B(v_j), q(v_j), \Pi(.|v_j)$. Then, if $j = N_v$, set $v_H(k) = v_{N_v}$, move to the next k and return to step 1. Otherwise, increment j (i.e. set $j = j + 1$) and return to step 5.

To construct mixed strategy local borrowing equilibria, we iteratively compute all mixed strategy local borrowing equilibria which have n points involving mixing, i.e. where $n = |\{v \in [v_L, v_H] | \Pi(v|v) \in (0,1)\}|$. This strategy is exhaustive because every local borrowing equilibrium with $n+1$ points involving mixing agrees below its highest mixing point with some local borrowing equilibrium with n mixing points. This iterative procedure takes as an input some list of local borrowing equilibria:

$$\{v_L(i), v_H(i), v_{mix}^{max}(i), B(v|i), q(v|i), g(v|i), \Pi(v|i)\}_{i\in\{1,N_{beqm}\}}$$

where $v_{mix}^{max}(i)$ is given by:

$$v_{mix}^{max}(i) = \begin{cases} v_L(i) & \forall v \in [v_L(i), v_H(i)]\, \Pi(v|i) \in \{0,1\} \\ max\{\{v|v \in \{[v_L(i), v_H(i)]\}, \Pi(g(v)|v) \in (0,1)\}\} & \text{otherwise} \end{cases}$$

This input list is initialized as the full list of pure strategy local borrowing equilibria. Initialize an empty output list. Also initialize a master list as the full list of pure strategy local borrowing equilibria. Then, for each equilibrium i in the input list:

1. For each $v_k \in (v_{min}^{max}(i), v_H(i)]$, calculate the stationary value and price, $B^{st}(v_k)$ and $q^{st}(v_k)$, respectively:

$$q^{st}(v_k) = \frac{F(v_k)(r+\delta)}{R-(1-\delta)F(v_k)} \tag{15}$$

$$B^{st}(v_k) = \frac{R-F(v_k)+\delta F(v_k)}{r+\delta}\frac{y-c(v_k,v_k)}{R-F(v_k)} \tag{16}$$

2. Note that we always have $q^{st}(v_k) > q(v_k|i)$ For the same set of v_k values, check whether $B^{st}(v_k) \geq B(v_k|i)$.

3. For every v_k satisfying this condition, calculate the $q_{mix}(v_k) \in [q(v_k|i), q^{st}(v_k)]$ and associated probability $\pi(v_k)$ of choosing $g(v_k|i)$ such that $\hat{B}(v_k, q_{mix}) = B(v_k)$.

4. Then, for each such v_k, create a new local equilibrium which has $v_L = v_L(i)$ and the equilibrium object values for as local equilibrium i for $v < v_k$. At v_k, set $g^{new}(v_k) = g(v_k|i), B^{new}(v_k) = B(v_k|i), q^{new}(v_k) = q_{mix}(v_k), \Pi(.|v_k) = \pi(v_k)$.

5. For each "new local equilibrium" begun above, set $j = k+1$ and go to step 5 of the local pure strategy borrowing equilibrium algorithm. Add the result to the "new list".

If the output list contains no elements at all, exit. Otherwise, add the output list to the master list and pass the output list (along with an empty output list for the next iteration) as the input list of above algorithm.

In order to construct the set of possible connections, at every point v in the grid, we find all the local equilibria that have v in their domain. For each such local equilibrium, we find every other local equilibrium with a point adjacent to v in its domain. A connection is possible at v for the pair of equilibria in question if any of the following conditions is true:

1. $v = v_L, g(v_L) = v_L = v'_H = g'(v'_H)$ (i.e. a local borrowing and savings equilibrium associated with the same stationary point).

2. $v = v_L, v'_L < v_L, g(v_L) = v_L,\ g'(v_{L,-}) \leq v_{L,-})$ and both

   (a) $B'(v_{L,-}) \geq \frac{y-c(v_{L,-},v_L)+q(v_L)B(v_L)}{r+\delta+(1-\delta)q(v_L)}$

   (b) and $B(v_L) \geq max_{v' \in [v'_L, v_{L,-}]} \frac{y-c(v_L,v')+q'(v')B'(v')}{r+\delta+(1-\delta)q'(v')}$

   hold (i.e. a local borrowing equilibrium joined, at its associated stationary point, to another local borrowing equilibrium, and at the points "adjacent" to the resulting connection, the policies associated with each original equilibrium remain optimal).

3. $v = v_{H,+}, v'_H > v_H, g(v_H) = v_H,\ g'(v_{H,+}) \geq v_{H,+})$ and both

   (a) $B(v_{H,+}) \geq \frac{y-c(v_{H,+},v_H)+q(v_H)B(v_H)}{r+\delta+(1-\delta)q(v_H)}$

   (b) and $B(v_H) \geq max_{v' \in [v_{H,+}, v'_H]} \frac{y-c(v_H,v')+q'(v')B'(v')}{r+\delta+(1-\delta)q'(v')}$

   hold (i.e. a local savings equilibrium joined, at its associated stationary point, to another local savings equilibrium, and at the points "adjacent" to the resulting connection, the policies associated with each original equilibrium remain optimal).

4. $v \in \{v'_L, ..., v'_H\},\ v_H \geq v'_L,\ g(v_L) = v_L,\ g'(v'_H) = v'_H$, and both

   (a) $B'(v) \geq max_{v' \in [v_L, v_-]} \frac{y-c(v,v')+q(v')B(v')}{r+\delta+(1-\delta)q(v')}$

   (b) and $B(v_-) \geq max_{v' \in [v, v'_H]} \frac{y-c(v_H,v')+q'(v')B'(v')}{r+\delta+(1-\delta)q'(v')}$

   hold (i.e. a local savings equilibrium joined to a local borrowing equilibrium associated away from their stationary points, and at the points "adjacent" to the resulting connection, the policies associated with each original equilibrium remain optimal).

where with some abuse of notation $v_{i,+}$ refers to the smallest point in the grid strictly greater than v_i and $v_{i.-}$ refers to the largest point in the grid strictly less than v_i. Since the conditions for joining two local equilibria depend only on the values (or potential values) of their objects at points adjacent to the connection, knowing the entire set of possible pairings is sufficient to construct all possible compatible collections of more than two local equilibria, i.e. all equilibria of the full model.

With the full list of local equilibria and possible connections in hand, we start at one edge of the state space with a working partial equilibrium list containing every local equilibrium that has v_H equal to the upper bound of the v grid. Initializing $k = N_v - 1$, we proceed thereafter as follows:

1. Initialize a new partial equilibrium list.
2. For each partial equilibrium in the working list, check whether the component local equilibrium active at $k+1$ can be connected to other local equilibrium at v_{k+1} (and therefore the first point associated with values of the connected equilibrium would be v_k). Add the result of all such possible connections to the new partial equilibrium list.
3. For each partial equilibrium in the working list, check whether the component local equilibrium active at $k+1$ has $v_L \leq v_k$. If it does, add it in the new partial equilibrium list.
4. Replace the working list with the new list. Deincrement k and return to step 1 (unless $k = 1$, in which case, exit).

The result will be the full set of equilibrium (q, B) associated with the discretized model and a g, Π consistent with each.

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