Generalized Delegation*

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November 2025

Abstract

We provide a general sufficient theorem to solve the delegation problem with and without money burning. For interval allocations, our main theorem relaxes the conditions obtained in previous literature. Significantly, our theorem extends beyond this, successfully addressing cases with discontinuous allocations. We demonstrate the theorem's power through examples, connecting with existing results and uncovering novel optimal allocation structures.

1 Introduction

In the standard delegation problem, a principal must decide how much discretion to grant to an agent who chooses an action. The trade-off confronting the principal is that the agent has superior information about the state of the world but is biased (i.e., does not share the principal's preferences). A defining feature of the delegation problem is that the principal is unable to use contingent transfers to shape the agent's incentives; thus, in the standard formulation, the principal selects a set of permissible actions from the real line, and the agent picks an action from this set. For example, the principal may utilize "interval delegation" whereby the set of permissible actions is an interval. Alternatively, the set of permissible actions may contain only a finite number of distinct actions. Hybrid sets that include distinct actions and intervals are also possible.

The delegation model begins with Holmström (1977) and is now a workhorse model in economics. The applications of the delegation framework are wide-ranging and include consumption-savings (Amador, Werning, and Angeletos, 2006), regulation (Alonso and Matouschek, 2008; Amador and Bagwell, 2022; Kolotilin and Zapechelnyuk, 2024), veto bargaining (Mylovanov,

*We thank Navin Kartik for comments.

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2008; Kartik, Kleiner, and Van Weelden, 2021), tariff caps (Amador and Bagwell, 2013), fiscal rules (Halac and Yared, 2014, 2022; Sublet, 2023), and quality certification (Zapechelnyuk, 2020), for example. Given this breadth of applications, a central goal for the delegation literature has been to determine general sufficient conditions under which interval delegation is optimal among all incentive-compatible allocations. Melumad and Shibano (1991), Alonso and Matouschek (2008) and Amador and Bagwell (2013) provide findings of this kind.

Amador and Bagwell (2013) analyze the optimality of interval delegation in the context of a general representation of the delegation problem.[1] The agent's action is taken from an interval on the real line, the state has a continuous distribution over a bounded interval on the real line, the principal's payoff function is continuous in the action and state and twice differentiable and concave in the action, and the agent's welfare function is twice differentiable and concave in the action with the state entering this function in a standard multiplicative fashion. Using and extending the Lagrangian methods developed by Amador, Werning, and Angeletos (2006), Amador and Bagwell provide sufficient conditions for optimal delegation to take the form of interval delegation. They also identify a preference family under which their sufficient conditions are necessary for the optimality of interval delegation.

Although the applications for interval delegation are extensive, several important applications feature other forms of delegation. As Saran (2024) discusses, volunteer organizations, such as hospitals, can offer volunteers only predetermined shifts, and charities can have donation campaigns that feature a finite menu of suggested donation amounts. The resulting allocations are discontinuous with respect to the agent's information or type. Sufficient conditions for the optimality of discontinuous allocations are less well understood, although the early work by Melumad and Shibano (1991) provides results for a specific setting (quadratic preferences, uniform distribution). As discussed in the following, Saran's recent work also provides sufficient conditions.

In this paper, we extend the Lagrangian approach and provide general sufficient conditions for optimal delegation. We capture Amador and Bagwell (2013) sufficient conditions for the optimality of interval delegation as a special case, but crucially, our analysis is not limited to settings in which interval delegation is optimal. We also provide sufficient conditions for the optimality of incentive-compatible allocations that are discontinuous. Thus, for example, we provide sufficient conditions for the optimality of incentive-compatible allocations that are induced when the permissible set is comprised of a finite number of actions. In addition, we provide sufficient conditions for the optimality of incentive-compatible hybrid allocations that arise when the permissible set of actions features a combination of distinct actions and intervals. The key methodological

[1] Amador and Bagwell report their results both when money burning (that is, throwing away resources) is infeasible, as in the standard delegation problem, and when money burning is allowed. For conditions under which optimal delegation entails actual money burning, see Ambrus and Egorov (2017) and Amador and Bagwell (2020).

innovation is extending the Lagrangian approach to handle discontinuous allocations through carefully constructed multiplier functions.

To develop these findings, we begin by considering "escalators-and-stairs" allocations. An escalators-and-stairs allocation partitions the space into a finite number of intervals. Within each interval, either (i) all types are allocated the the same action (pooling), or (ii) each type receives their preferred or "flexible" action (separation). Restating a result by Melumad and Shibano (1991), we show that an escalators-and-stairs allocation is incentive-compatible if and only if it satisfies the properties that (i) adjacent pooling intervals entail an upward jump at the indifferent type, and (ii) adjacent intervals such that one separates and one pools are continuous with the pooling action thus corresponding to the flexible action for the boundary type.

We next characterize the necessary conditions for an escalators-and-stairs allocation to be optimal within a restricted class of incentive-compatible escalators-and-stairs allocations. The restricted class shares a common partition with the same intervals of separation and pooling. We associate these restrictions with a finite set of scalar multipliers and characterize the first-order conditions using these multipliers.

Armed with these results, we then proceed to our final step and characterize in Theorem 1 sufficient conditions under which a given incentive-compatible escalators-and-stairs allocation is optimal among the class of all incentive-compatible allocations. To do so, and following Amador, Werning, and Angeletos (2006) and Amador and Bagwell (2013), we construct (cumulative) multiplier functions where the independent variable is the type. As we show, these multiplier functions can be defined in terms of scalar multipliers for the restricted problem just described. To apply Theorem 1, one simply presents a candidate solution that is an incentive-compatible escalators-and-stairs allocation. For the associated pooling and separation intervals, the sufficient conditions then indicate the properties to be satisfied by the welfare functions of the principal and agent and also the distribution function. Importantly, our sufficient conditions can thus be used to address the optimality of interval allocations, pooling allocations that entail a finite number of distinct actions, and hybrid allocations that include both intervals of pooling and separation.

We also use Theorem 1 to characterize sufficient conditions for the optimality of various specific allocations, including those featured in the previous literature. We show that Theorem 1 relaxes Amador and Bagwell (2013) sufficient conditions for the optimality of interval delegation. An interval allocation is an escalators-and-stairs allocation with an interval partition of at most three regions, where in the case of three regions, the middle region entails separation, and the exterior regions involve pooling (at the floor and cap, respectively). We also consider the optimality of an extreme one-step allocation, which in our context is an escalators-and-stairs allocation that allows for exactly two actions and thus induces two pooling regions with an upward jump at the indifferent type. Melumad and Shibano (1991) provide sufficient conditions for the optimality of this allocation in linear-quadratic payoffs and uniformly distributed types. We show that Theorem 1 also provides these sufficient conditions as a special case. As a further illustration of the reach of Theorem 1, we consider a specific hybrid example in which the optimal allocation has both separating and adjacent pooling regions. Finally, we consider an extension of the hybrid example to preferences that fall outside the family of preferences identified by Amador and Bagwell (2013) and emphasized in subsequent work, thus confirming that the applicability of Theorem 1 extends beyond this preference family.

Three recent contributions are important to mention. Kolotilin and Zapechelnyuk (2024) present a clever argument that maps the delegation problem (with or without a participation constraint) to an optimal persuasion problem. Kleiner, Moldovanu, and Strack (2021) show how to use powerful tools from majorization to solve the delegation problem. These two papers have significantly expanded the set of tools available to tackle delegation problems.[2] Our paper complements and provides an alternative approach. Relying on our Lagrangian approach, we uncover similar results as these papers do, but as noted in certain cases, we can go beyond the family of preferences that both of these papers rely on.[3] Finally, Saran (2024) considers a general delegation problem with a participation constraint and establishes conditions under which the optimal allocation has a finite number of jumps. He also numerically solves for the optimal allocation under specific utility functions. Our approach differs in that we use a Lagrangian approach to provide general analytical characterizations of optimal allocations for a setting without a participation constraint.

As a further extension of our analysis, we extend our sufficient condition to the case where money burning is feasible. In this way, we establish the optimality of escalators-and-stairs allocations even when money burning is feasible. The extension of our results follows the approach used by Amador and Bagwell (2013) in establishing conditions for the optimality of interval allocations when money burning is feasible.

The remainder of the paper is structured as follows. Section 2 presents the model. Sections 3 and 4 present the types of allocations that are incentive-compatible (escalators-and-stairs) and discuss the necessary conditions. Section 5 presents the sufficient conditions and our main result, Theorem 1. Section 6 discusses the results and relates our findings to the literature. Section 7

[2]Both of these papers allow for stochastic mechanisms. Kartik, Kleiner, and Van Weelden (2021) in their analysis of delegation in veto bargaining also study stochastic mechanisms. The use of stochastic mechanisms is related to the value of money burning in a delegation setting. For previous work featuring money-burning considerations, see for example Amador, Werning, and Angeletos (2006), Amador and Bagwell (2013), Ambrus and Egorov (2017), and Amador and Bagwell (2020).

[3]Kleiner (2022) is also relevant. He extends the delegation literature to multidimensional settings under the assumption that stochastic mechanisms are available. Our approach is again complementary and utilizes an alternative Lagrangian approach. For the single-dimensional setting, we also report results that go beyond the family of preferences assumed by Kleiner.

concludes.

2 Model

We consider a standard delegation problem with a principal and an agent allowing for the possibility of money burning. The principal has a payoff function given by $w(\gamma, \pi) - t$, while the agent has a payoff given by $\gamma\pi + b(\pi) - t$. The value of π represents an action or allocation, and the value of γ represents a state or shock that is private information to the agent. The value of t represents a money-burning action which generates losses for both parties that we normalize to be equal without loss of generality.

The agent observes the state or type $\gamma \in \Gamma \equiv [\underline{\gamma}, \overline{\gamma}]$ with $\overline{\gamma} > \underline{\gamma}$ drawn from a continuous distribution F with an associated continuous density f. The action π is chosen by the agent from a feasible set $\Pi = [0, \overline{\pi}]$ with $\overline{\pi} > 0$ denoting the maximum possible action. For the remainder of the paper, we impose the following conditions.

Assumption 1. *The following holds: (i) the function $w : \Gamma \times \Pi \to \mathbb{R}$ is continuous on $\Gamma \times \Pi$; (ii) for any $\gamma_0 \in \Gamma$, the function $w(\gamma_0, \cdot)$ is concave on Π, and twice differentiable on $(0, \overline{\pi})$; (iii) the function $b : \Pi \to \mathbb{R}$ is strictly concave on Π, and twice differentiable on $(0, \overline{\pi})$; (iv) there exists a twice differentiable and strictly increasing function $\pi^f : \Gamma \to (0, \overline{\pi})$ such that $\pi^f(\gamma_0) \in \arg\max_{\pi\in\Pi} \{\gamma_0\pi + b(\pi)\}$ for all $\gamma_0 \in \Gamma$; and (v) the function $w_\pi : \Gamma \times (0, \overline{\pi}) \to \mathbb{R}$ is continuous on $\Gamma \times (0, \overline{\pi})$, where w_π denotes the derivative of w in its second argument.*

Note that $\pi^f(\gamma)$ denotes the agent's preferred action given the private information γ. Assumption (iv) restricts attention to preferences such that π_f is strictly increasing in γ.

In the delegation problem, the principal does not observe the realization of the type γ. However, the principal can restrict the choice set of the agent. As is standard, we write the principal's problem as a mechanism design problem. The principal chooses an allocation $(\boldsymbol{\pi}, \boldsymbol{t})$ which is a pair of functions $\boldsymbol{\pi} : \Gamma \to \Pi$ and $\boldsymbol{t} : \Gamma \to \mathbb{R}$ that determine the action and the amount of money burned given the private information of the agent. The goal is to choose $(\boldsymbol{\pi}, \boldsymbol{t})$ so as to maximize the principal's welfare function subject to the incentive constraints of the agent:

$$\max_{\{\boldsymbol{\pi},\boldsymbol{t}\}} \int_{\Gamma} (w(\gamma, \boldsymbol{\pi}(\gamma)) - \boldsymbol{t}(\gamma)) f(\gamma) d\gamma \quad \text{subject to:} \tag{1}$$
$$\gamma \in \arg\max_{\tilde{\gamma}\in\Gamma} \{\gamma \boldsymbol{\pi}(\tilde{\gamma}) + b(\boldsymbol{\pi}(\tilde{\gamma})) - \boldsymbol{t}(\tilde{\gamma})\}$$
$$\boldsymbol{t}(\gamma) \geq 0, \forall \gamma \in \Gamma$$

where the constraint is an incentive-compatibility constraint that arises since the agent is privately informed of the value of γ. It follows from the supermodularity of the agent's payoff that an incentive-compatible allocation $\boldsymbol{\pi}(\gamma)$ must be non-decreasing in γ. We refer to Problem (1) as the *problem with money burning*.

We also consider the problem where money burning is ruled out by assumption, which corresponds to the above problem with the additional constraint:

$$\boldsymbol{t}(\gamma) = 0, \forall \gamma \in \Gamma \tag{2}$$

We refer to Problem (1) with this additional constraint as the *problem without money burning*.

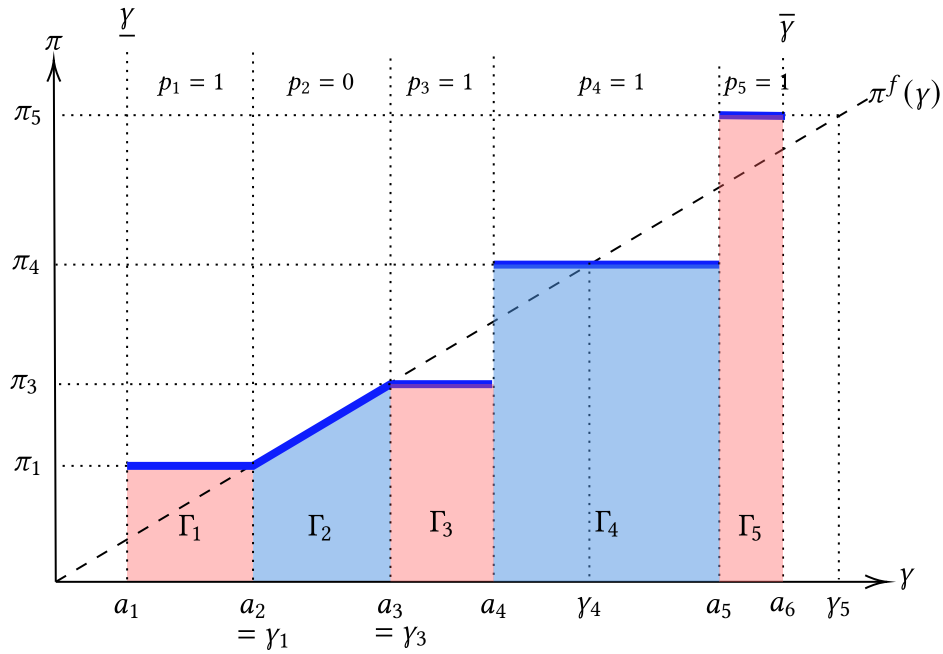
A particularly simple allocation is the "interval allocation." Amador and Bagwell (2013) obtain general sufficient conditions for interval delegation to be optimal with and without money-burning. This corresponds to a case where the optimal allocation does not burn any money and $\boldsymbol{\pi}^\star$ is defined by $\gamma_L < \gamma_H$ such that $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma)$ for $\gamma \in [\gamma_L, \gamma_H] \subset \Gamma$, $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma_L)$ for $\gamma < \gamma_L$, and $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma_H)$ for $\gamma > \gamma_H$. Our goal in this paper is to obtain sufficient conditions for other allocations that do not burn money to be optimal, including those that are not of the interval form.

3 Escalators and stairs

We now define the class of allocations we study. Consider an interval $\tilde{\Gamma} \subset \Gamma$. We say that an allocation $\boldsymbol{\pi}$ *separates* in $\tilde{\Gamma}$ if $\boldsymbol{\pi}(\gamma) = \pi^f(\gamma)$ for all $\gamma \in \tilde{\Gamma}$. We say that an allocation *pools at* π in $\tilde{\Gamma}$ if $\boldsymbol{\pi}(\gamma) = \pi$ for all $\gamma \in \tilde{\Gamma}$.

An allocation $\boldsymbol{\pi}$ lies within the "escalators-and-stairs" class if there exists a finite partition of Γ, $\{\Gamma_i\}_{i=1}^n$, with $\Gamma_i \equiv [a_i, a_{i+1}) \subset \Gamma$ for $i \in \{1, \ldots, n-1\}$ and $\Gamma_n \equiv [a_n, a_{n+1}] \subset \Gamma$ and $a_i < a_{i+1}$, such that $\cup_{i\in\{1,\ldots,n\}} \Gamma_i = \Gamma$ and for every $i \in \{1, \ldots, n\}$, $\boldsymbol{\pi}$ either *separates* or *pools* in Γ_i. It is without loss of generality to consider allocations within this class such that no two adjacent intervals are separating, and no two adjacent intervals are pooling at the same action (otherwise, we could merge them into one). We maintain these requirements in what follows.

Figure 1: An escalators-and-stairs allocation.



Note: The p_i notation showcased in this figure is explained in Section 4.

Absent money burning, incentive compatibility imposes additional restrictions on an "escalators-and-stairs" allocation. In particular, consider the following condition:

Condition 1. *An escalators-and-stairs allocation* $\boldsymbol{\pi}$ *with the associated interval partition* $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ *satisfies:*

(i) For any two adjacent intervals Γ_i, Γ_{i+1} *that are pooling at* π_i, π_{i+1} *respectively,* $\pi_i < \pi_{i+1}$ *and* $\gamma\pi_i + b(\pi_i) = \gamma\pi_{i+1} + b(\pi_{i+1})$ *for* $\gamma = a_{i+1}$.

(ii) For any two adjacent intervals Γ_i, Γ_{i+1} *such that one of them separates and the other one pools at* π*, it must be that* $\pi = \pi^f(\gamma)$ *where* $\gamma = a_{i+1}$.

Note that part (i) of Condition 1 necessarily requires that for two adjacent intervals that pool at π_i and π_{i+1}, respectively, it must be $\pi_i < \pi^f(\gamma) < \pi_{i+1}$ for $\gamma = a_{i+1}$.

Figure 1 illustrates an escalators-and-stairs allocation. For the types in Γ_1, the allocation requires pooling at the action π_1. Separation is then specified for the types in Γ_2 with each type thus selecting its flexible action, which, as the figure suggests, can be envisioned as moving up an "escalator." The allocation then requires pooling at the action π_3 for types in Γ_3. As required by part (ii) of Condition 1, notice that the allocation is continuous at the lower and upper boundaries of the separating region Γ_2. In contrast, and as required by part (i) of Condition 1, a jump in the allocation occurs at type a_4, with this type being indifferent between pooling at π_3 and π_4. As illustrated in the figure, a jump between two pooling regions gives the appearance of moving up "stairs." For the types in Γ_4, observe that the pooling action π_4 is above (below) the flexible action for lower (higher) types. Finally, another upward jump in the illustrated allocation occurs at type a_5, with type a_5 being indifferent between pooling at π_4 and π_5.

A first result, a restatement of Melumad and Shibano (1991) in our notation, states that Condition 1 is necessary and sufficient for incentive compatibility:

Proposition 1. *An escalators-and-stairs allocation $\boldsymbol{\pi}$, with $\boldsymbol{t} = 0$, is incentive-compatible if and only if it satisfies Condition 1.*

Proof. Although Melumad and Shibano (1991) already proved this, we provide a proof in Appendix B.2 for completeness. □

Before moving on to characterize sufficient conditions for an escalators-and-stairs allocation to be optimal within the set of all incentive-compatible allocations, it is helpful first to focus on optimality within the set of incentive-compatible escalators-and-stairs allocations.[4]

4 Optimality within escalators-and-stairs

In this section, we derive the necessary conditions that must be satisfied for an escalators-and-stairs allocation to be optimal within a certain subset of all incentive-compatible escalators-and-stairs allocations.

Given an interval partition $\{\Gamma_i\}_{i=1}^n$, we define the vector $p = \{p_i\}_{i=1}^n$ to be such that $p_i = 1$ if the allocation pools in Γ_i and $p_i = 0$ otherwise. We denote as π_i the action level at which the allocation pools in Γ_i and let $\pi_1 = \boldsymbol{\pi}(\underline{\gamma})$.

Recalling that no two adjacent intervals can separate, we let

$$P_n \equiv \{p \in \{0,1\}^n | \text{ such that } p_i = p_{i+1} = 0 \text{ for no } i\}.$$

[4]It is possible to show that an incentive-compatible escalators-and-stairs allocation is uniquely determined by knowing how the allocation separates and pools in a partition and the action assigned to the lowest type if there is no separating interval. This is shown in Lemma 2 in the Appendix.

Then we have the following requirements for a valid partition:

$$a_1 = \underline{\gamma};\, a_{n+1} = \overline{\gamma};$$
$$\forall i \in \{1, ..., n\} : \quad \begin{cases} a_i \in \Gamma;\, a_{i+1} > a_i; \text{ and} \\ \pi_i \in \Pi \quad \text{ if } p_i = 1. \end{cases} \tag{3}$$

Together with Condition 1, incentive-compatible escalators-and-stairs allocation are then determined by the following restrictions:

$$\forall i \in \{1, ..., n-1\} : \quad \begin{cases} \pi_i = \pi_f(a_{i+1}), & \text{if } p_i = 1, p_{i+1} = 0; \\ \pi_f(a_{i+1}) = \pi_{i+1}, & \text{if } p_i = 0, p_{i+1} = 1; \\ a_{i+1}\pi_i + b(\pi_i) = a_{i+1}\pi_{i+1} + b(\pi_{i+1}), & \text{if } p_i = p_{i+1} = 1; \\ \pi_{i+1} > \pi_i, & \text{if } p_i = p_{i+1} = 1; \end{cases} \tag{4}$$

The constraints in the first line of (3) are just that a_i represents a valid interval bound, that the subsets of the partition be nonempty, and that π_i be feasible. The constraints in the first and second lines of (4) are part (ii) of Condition 1, which applies to pooling and separating regions that are adjacent. The next two lines are part (i) of Condition 1.

Given $\{a_i, \pi_i\}_{i=1}^n$ and a vector $p \in P_n$, the objective of the principal can then be written as

$$P(\{a_i, \pi_i\}_{i=1}^n | p) \equiv \sum_{i=1}^n \left(p_i \int_{a_i}^{a_{i+1}} w(\gamma, \pi_i) f(\gamma) d\gamma + (1 - p_i) \int_{a_i}^{a_{i+1}} w(\gamma, \pi_f(\gamma)) f(\gamma) d\gamma \right),$$

where we have re-arranged the expected payoff to the principal by first summing over the payoff in the pooling subsets, and then adding the payoff in the separating subsets.

An escalators-and-stairs allocation that is optimal within the set of all incentive-compatible allocations must also be optimal within the set of escalators-and-stairs allocations that share partitions with the same number of subsets and the same vector p. Thus, an optimal escalators-and-stairs allocation must be a solution to

$$\max_{\{a_i, \pi_i\}_{i=1}^n} P(\{a_i, \pi_i\}_{i=1}^n | p) \text{ subject to (3) and (4).} \tag{5}$$

For a given action level π_i, it is useful to define γ_i as $\gamma_i \equiv -b'(\pi_i)$. If $\pi_i \in [\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma})]$, then γ_i corresponds to the type in Γ such that $\pi_i = \pi^f(\gamma_i)$, i.e. the type at which π_i coincides with the agent's preferred action. This naturally leads to the following necessary condition:

Condition 2 (Necessary Condition). *There exists* $\{\lambda_i\}_{i=1}^{n+1} \in \mathbb{R}$ *such that for all* $p_i = 1$ *(pools):*

(i) $\displaystyle\int_{a_i}^{a_{i+1}} w_\pi(\gamma, \pi_i) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1} - \gamma_i) - \lambda_i(a_i - \gamma_i).$

(ii) $\lambda_{i+1} = \left(\dfrac{w(a_{i+1}, \pi_{i+1}) - w(a_{i+1}, \pi_i)}{\pi_{i+1} - \pi_i} \right) f(a_{i+1})$ *if* $p_{i+1} = 1$.

with $\lambda_1 = \lambda_{n+1} = 0$.

Condition 2 part (i) indicates that, in a pooling region, the optimal choice of π balances two effects. The first is the direct marginal effect on payoffs for types in the region; this corresponds to the integral on the left hand side. The second corresponds to the effects of this change in adjacent intervals. The values of λ on the right-hand side represent the Lagrange multipliers of the incentive compatibility constraints for these intervals. If an adjacent interval is separating, then the incentive compatibility constraint holds automatically. To see this, recall from our definition of γ_i that $\gamma_i = -b'(\pi_i)$, and thus, if an adjacent interval is separating, then either $a_{i+1} = \gamma_i$ or $a_i = \gamma_i$. If an adjacent interval is not separating, then the corresponding value of λ is pinned down by the value in part (ii).

We can now state the following lemma.

Lemma 1. *Given* $p = \{p_i\}_{i=1}^n \in P_n$, $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ *solves Problem 5 only if* $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ *satisfies* (3) *and* (4) *and Condition 2 holds.*

Proof. In Appendix B.3. □

5 Sufficient conditions for optimality

Next, we will determine the conditions under which an allocation with interval partition $\{\Gamma_i\}_{i=1}^n$ and vector p solves Problem (1) with and without constraint (2).

Related to Amador and Bagwell (2013), for the problem without money burning, that is Problem (1) with the additional constraint (2), we define $\kappa(\gamma)$ as

$$\kappa(\gamma) \equiv \inf_{\pi \in \Pi} \left\{ \frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)} \right\}, \tag{6}$$

for all $\gamma \in \Gamma$.

For the problem with money burning, that is Problem (1), we define $\kappa(\gamma)$ as

$$\kappa(\gamma) \equiv \min\left\{\inf_{\pi\in\Pi}\left\{\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}\right\}, 1\right\}, \tag{7}$$

for all $\gamma \in \Gamma$.

Given a function $\kappa(\gamma)$, we define the following function $G(\gamma)$:

$$G(\gamma) \equiv \int_{\underline{\gamma}}^{\gamma} \kappa(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma}. \tag{8}$$

We will assume that this function is well defined.[5]

We are now ready to state the main sufficient condition in the paper.

Condition 3 (Sufficient Condition)**.** *Given the escalators-and-stairs allocation* $\boldsymbol{\pi}$ *with interval partition* $\{\Gamma_i\}_{i=1}^{n}$ *there exists* $\{\lambda_i\}_{i=1}^{n+1}$ *with* $\lambda_1 = \lambda_{n+1} = 0$ *such that*

(c1) If $p_i = 0$*, then*

$$G(\gamma) - w_\pi(\gamma, \pi_f(\gamma)) f(\gamma)$$

is non-decreasing $\forall \gamma \in \Gamma_i$*.*

(c2) If $p_i = 1$ *then*

$$\int_{a_i}^{a_{i+1}} w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) d\tilde{\gamma} = \lambda_{i+1}(a_{i+1} - \gamma_i) + \lambda_i(\gamma_i - a_i), \tag{9}$$

for $\gamma \in \Gamma_i \cap [a_i, \gamma_i)$*:*

$$\int_{a_i}^{\gamma} \left[w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) - \lambda_i\right] d\tilde{\gamma} \geq (\gamma - \gamma_i)\left[G(\gamma) - G(a_i)\right], \tag{10}$$

for $\gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}]$*:*

$$\int_{\gamma}^{a_{i+1}} \left[w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) - \lambda_{i+1}\right] d\tilde{\gamma} \leq (\gamma - \gamma_i)\left[G(a_{i+1}) - G(\gamma)\right], \tag{11}$$

and if $a_i < \gamma_i < a_{i+1}$*:*

$$G(a_i) - \lambda_i \leq G(a_{i+1}) - \lambda_{i+1} \tag{12}$$

[5] In Amador and Bagwell (2013), κ was constrained to be a constant, independent of γ. Our new definition of $\kappa(\gamma)$ relaxes their results even when applied to interval delegations.

(c3) If $p_1 = 0$, or $p_1 = 1$ and $\gamma_1 = a_1$, then $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) \leq 0$.

(c4) If $p_n = 0$, or $p_n = 1$ and $\gamma_n = a_{n+1}$, then $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma}))f(\overline{\gamma}) \geq 0$.

where $p = \{p_i\}_{i=1}^n \in P_n$ is the vector associated with the pooling indicators, $\{a_i, \pi_i\}_{i=1}^n$ is the pair of vectors associated with the lower bounds of the interval and the pooling actions, and G is defined by (8) given κ.

Condition 3 contains several parts. Part (c1) applies for intervals where the allocation separates. Part (c2) applies in intervals where the allocation is pooling. And parts (c3) and (c4) apply to the lowest and highest regions in the partition.

With this, we can now state the main result:

Theorem 1 (Sufficiency)**.** *Consider an incentive-compatible escalators-and-stairs allocation $\boldsymbol{\pi}$ with associated partition $\{\Gamma_i\}_{i=1}^n$.*

(a) (No money burning) If Condition 3 is satisfied with $\kappa(\gamma)$ given by (6), then the allocation $\boldsymbol{\pi}$ solves the problem without money burning, that is, Problem (1) with the additional constraint (2).

(b) (Money burning) If Condition 3 is satisfied with $\kappa(\gamma)$ given by (7), then the allocation $\boldsymbol{\pi}$ solves the problem with money burning, that is, Problem (1).

Proof. In Appendix B.4. □

To prove this theorem, we follow and extend the Lagrangian approach developed by Amador and Bagwell (2013). Consider the case with money burning. Our proof proceeds through five steps. First, we reformulate the agent's incentive constraints in integral form, along with a monotonicity restriction. Since money burning must be non-negative, this introduces an additional inequality constraint for all $\gamma \in \Gamma$. Second, we propose a Lagrange multiplier function for this constraint. Given an escalators-and-stairs allocation, we must establish three properties of this multiplier: (i) it is non-decreasing, (ii) it satisfies complementary slackness, and (iii) the proposed allocation maximizes the resulting Lagrangian over all non-decreasing allocation rules. Third, we apply Condition 3 to verify that our proposed multiplier is indeed non-decreasing. Fourth, we establish that the Lagrangian is concave over the set of non-decreasing allocations $\boldsymbol{\pi}$, which ensures the validity of the first order approach we use next. Finally, building on Amador, Werning, and Angeletos (2006), we derive the first-order conditions for the Lagrangian maximization problem and verify that these conditions are satisfied at our proposed escalators-and-stairs allocation.

Differently from both papers, here we have to deal with a possible discontinuity of the allocation $\boldsymbol{\pi}$, which corresponds to points where the allocation jumps between two pooling regions. We solve this by proposing a multiplier that is continuous at these jump points in the allocation. The proposed multiplier is however not continuous everywhere, and has discontinuities at $\{a_i\}_{i=1}^{n+1}$ and potentially at the γ_i points. These jumps must be positive, as the multiplier must be non-decreasing. The main arguments in the proof show that Condition 3 guarantees that such a multiplier can be constructed.

The case without money burning is very similar. The main difference is that, as in Amador and Bagwell (2013), the no money burning constraint requires imposing two inequality conditions with two associated Lagrange multipliers. The conditions of the theorem again guarantee that such multipliers can be constructed.

6 Discussion of results

Theorem 1 shows that Condition 3 is sufficient for an escalators-and-stairs allocation to be optimal within the set of all incentive-compatible allocations.

Before moving on, it may be helpful to discuss the role of money burning in the analysis above. Suppose that the preferences of the principal can be written as:

$$w(\gamma, \pi) = \theta \tilde{w}(\gamma, \pi).$$

for some scaling parameter θ.

Without money burning, changes in θ have no effect on Condition 3, as both κ, w, and G (and the corresponding multipliers $\{\lambda_i\}$) scale proportionally with θ. This is as expected: θ is just a scale parameter.

However, in the presence of money burning, an increase in θ is no longer innocuous. To see this, note that we can rewrite the principal's preferences as

$$\theta\Big(\tilde{w}(\gamma, \pi) - t/\theta\Big)$$

and thus a higher value of θ effectively makes money burning *less costly* to the principal. An increase in θ, say from a benchmark value of 1, no longer scales G in the same manner as w, as κ is capped from above by 1. The sufficient condition then becomes harder to satisfy.

As a final point, it is interesting to note that if

$$\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)} \leq 1 \tag{13}$$

for all γ, then the sufficient condition is the same whether money burning is allowed or not.

6.1 Necessary conditions for optimal delegation

Condition 3 provides sufficient conditions for global optimality. As a consistency check, Condition 3 should imply Condition 2 (necessary conditions for a restricted problem). In this subsection, we verify that this is indeed the case, and that the values of $\{\lambda_i\}$ in Condition 3 can be used in Condition 2. This subsection also uncovers an additional preference restriction that is implied by Condition 3.

First note that if Condition 3 holds for some $\{\lambda_i\}_{i=1}^{n+1}$, then part (i) of Condition 2 holds for the same $\{\lambda_i\}_{i=1}^{n+1}$ given that (9) holds.

In Appendix A we show that Condition 3 can only be satisfied for $p_i = p_{i+1} = 1$ if:

$$\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} = \kappa(a_{i+1}) \tag{14}$$
$$\forall\, \pi \in [\pi_i, \pi_{i+1}] \text{ and } \Gamma_i, \Gamma_{i+1} \text{ s.t. } p_i = p_{i+1} = 1.$$

With this, in Appendix A we show that Condition 3 then implies part (ii) of Condition 2, as expected. [6]

No money burning. In the absence of money burning, (14) is immediately satisfied for the family of preferences with the property that $\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}$ is constant for all γ and π. That is, (14) is satisfied for the family of preferences such that the principal's payoff takes the form

$$w(\gamma,\pi) = A[b(\pi) + B(\gamma) + C(\gamma)\pi] \tag{15}$$

for some constant $A > 0$ and functions B and C. Following Kolotilin and Zapechelnyuk (2024), we refer to this family as the *linear family*.[7]

[6] A reader may ask themselves how the "marginal" conditions in Condition 3 can ever imply the "global" condition in Condition 2 part (ii). The answer is that Condition 3 implicitly requires preferences such that the marginal indeed implies the global.

[7] This family of preferences is identified by Amador and Bagwell (2013) and plays a special role in their analysis when they generate conditions that are also necessary (and not just sufficient) for interval delegation to be optimal. The preferences in Amador, Werning, and Angeletos (2006) fall into this class. This preference family is the one for which Kolotilin and Zapechelnyuk (2024) obtains sufficient and necessary results mapping delegation problems into persuasion problems. In their paper using majorization tools in economic applications, Kleiner, Moldovanu, and

The result in our theorem is more general than imposing this family from the get-go, as it only requires this "linearity" in preferences to occur at values of γ where the allocation discontinuously jumps and only then for a subset of values of π (the values of π within the jump in the proposed allocation).

Money burning. In the case with money burning, (14) requires that $w_{\pi\pi}(a_{i+1}, \pi)/b''(\pi)$ be a constant *weakly lower* than 1 for $\pi \in [\pi_i, \pi_{i+1}]$ (a result that follows from the definition of κ for the case with money burning). So for example, our theorem provides conditions for optimal allocations where the principal's preferences are given by

$$w(\gamma, \pi) = A(\gamma)[b(\pi) + B(\gamma) + C(\gamma)\pi]$$

where $A(\gamma)$ is restricted to be lower than 1 at the γ points where the allocation jumps and potentially unrestricted everywhere else (as long as the rest of the sufficient conditions are satisfied).

Both Kolotilin and Zapechelnyuk (2024) and Kleiner, Moldovanu, and Strack (2021) analyze cases without money burning, but where the allocation is allowed to be stochastic. It is well known that making the allocation stochastic is akin to introducing money burning, as the concavity of the payoff functions implies that this randomness generates a loss to both the principal and the agent. Given that we have normalized the money-burning cost in payoffs to be the same for both the principal and the agent, money burning corresponds to stochastic mechanisms only when $A(\gamma) = 1$ for all γ. Note that in this case, the inequality (13) holds for all γ. This implies that whether or not we allow for stochastic mechanisms does not alter the sufficient conditions.

However, as we discussed above, if $A(\gamma)$ were to be scaled up by a factor $\theta > 1$, then money burning would be *less costly* than a corresponding stochastic mechanism, and one would expect the sufficient Condition 3 to become harder to satisfy.[8]

This discussion makes clear that our theorem covers cases with money burning that are not contained in previous results in the literature.

6.2 The conditions for interval delegation in AB

Amador and Bagwell (2013) obtained sufficient conditions for an interval allocation to be optimal. We can represent an interval allocation as an escalators-and-stairs allocation with an interval partition of at most three regions, where one region separates, the others pool, and the two

Strack (2021) also argue that their approach can be extended to the delegation problem with these same preferences (see footnote 40). Kleiner (2022) also restricts attention to this preference family (with $B(\gamma) = 0$) to characterize optimal delegation in multidimensional settings.

[8]We provide such an example in Section 6.5.

regions that pool cannot be adjacent. The separating region corresponds to the interval of actions that are offered to the agent, while the pooling regions represent the constraint that actions outside the interval are not allowed.

Let us focus on the case where the interval allocation is generated by just an upper bound on the action. In this case, the partition contains just two intervals Γ_1 and Γ_2 where $p_1 = 0$ and $p_2 = 1$. In this case, $a_1 = \underline{\gamma}$, $a_3 = \overline{\gamma}$; and we let $\gamma_H = a_2$ (for comparison with Amador and Bagwell, 2013). Incentive compatibility implies $\pi_2 = \pi_f(\gamma_H)$, and it follows that $\gamma_2 = \gamma_H$.

We can now apply Condition 3 to this case. Part (c1) requires that

$$\int_{\underline{\gamma}}^{\gamma} \kappa(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma} - w_\pi(\gamma, \pi_f(\gamma)) f(\gamma) \text{ be non-decreasing in } \gamma \in [\underline{\gamma}, \gamma_H) \tag{16}$$

Part (c3) requires that

$$w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) f(\underline{\gamma}) \leq 0 \tag{17}$$

Part (c2) at $i = 2$ becomes

$$\int_{\gamma_H}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma_H)) f(\tilde{\gamma})) d\tilde{\gamma} = 0 \tag{18}$$

using that $\lambda_3 = 0$ and $a_2 = \gamma_2 = \gamma_H$.

By part (c2), inequality (11), we have

$$\int_{\gamma}^{\overline{\gamma}} \left\{ w_\pi(\tilde{\gamma}, \pi_f(\gamma_H)) f(\tilde{\gamma}) \right\} d\tilde{\gamma} \leq (\gamma - \gamma_H)[G(\overline{\gamma}) - G(\gamma)] \text{ for } \gamma \in \Gamma_i \cap (\gamma_H, \overline{\gamma}]. \tag{19}$$

We can summarize (18) and (19) with

$$(\gamma - \gamma_H) \int_{\gamma}^{\overline{\gamma}} \kappa(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma} \geq \int_{\gamma}^{\overline{\gamma}} w_\pi \left(\tilde{\gamma}, \pi_f(\gamma_H) \right) f(\tilde{\gamma}) d\tilde{\gamma} \text{ for } \gamma \in [\gamma_H, \overline{\gamma}], \tag{20}$$

with equality at $\gamma = \gamma_H$. The rest of the parts in Condition 3 do not apply.

We now show that conditions (16), (17) and (20) represent a relaxation of the conditions obtained in Amador and Bagwell (2013) for the optimality of a cap allocation. In Amador and Bagwell (2013), the conditions for this case are:

$$\hat{\kappa}F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma) \text{ be non-decreasing in } \gamma \in [\underline{\gamma}, \gamma_H), \tag{21}$$

$$(\gamma - \gamma_H)\hat{\kappa}(1 - F(\gamma)) \geq \int_\gamma^{\overline{\gamma}} w_\pi\left(\tilde{\gamma}, \pi_f(\gamma_H)\right) f(\tilde{\gamma})d\tilde{\gamma} \text{ for } \gamma \in [\gamma_H, \overline{\gamma}], \tag{22}$$

$$w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) \leq 0, \tag{23}$$

where $\hat{\kappa}$ in Amador and Bagwell (2013) was effectively defined as

$$\hat{\kappa} = \inf_{\gamma \in \Gamma} \kappa(\gamma).$$

It is straightforward to show that if conditions (21), (22), and (23) hold, then (16), (17) and (20) also hold. The converse is not true. This highlights that Theorem 1 contains the result of Amador and Bagwell (2013) as a special case and generalizes it.[9]

6.3 The conditions for extreme delegation in MS

Melumad and Shibano (1991) solve for the optimal allocations in the case where the shocks are uniformly distributed, the preferences are linear-quadratic, and the preferred actions are linear functions of the shock. In our notation, this corresponds to $F(\gamma) = \gamma \in [0, 1]$, $b(\pi) = -\frac{\pi^2}{2}$ and $w(\gamma, \pi) = b(\pi) + (a\gamma + k)\pi$, for some $a, k \in \mathbb{R}$. This implies $\kappa = 1$, $\pi_f(\gamma) = \gamma$, and $\gamma_i = -b'(\pi_i) = \pi_i$ for $p_i = 1$.

When $a > 2$ and $k \in (1 - a, 0)$, the optimal allocation does not correspond to an interval allocation. Instead, it corresponds to a one-step allocation. In particular, the corresponding action levels, π_1 and π_2, lie outside the agent's preferred action range, that is, $\pi_1 < 0$ and $\pi_2 > 1$. We refer to this case as an *extreme one-step allocation*. We can describe an extreme one-step allocation as an escalators-and-stairs allocation with exactly two regions, $\{\Gamma_1, \Gamma_2\}$, where $p_1 = p_2 = 1$, $\pi_1 < 0$ and $\pi_2 > 1$. Incentive compatibility requires $a_2 = (\pi_1 + \pi_2)/2$. For comparison with Melumad and Shibano (1991), let $a_1 = 0$, $a_3 = 1$, and $a_2 = \tau \in (0, 1)$.

We can now apply the relevant parts of Condition 3, which correspond to conditions (9) and (11) for $p_1 = 1$ and conditions (9) and (10) for $p_2 = 1$. The equality conditions in (9) imply that:

[9] A similar approach can be used for the other two interval allocation cases (i.e., the allocation that corresponds to just a lower bound on the action and the one that corresponds to both a cap and a floor).

$$\int_0^\tau (a\tilde{\gamma} + k - \pi_1) d\tilde{\gamma} = \lambda_2(\tau - \pi_1) \tag{24}$$

$$\int_\tau^1 (a\tilde{\gamma} + k - \pi_2) d\tilde{\gamma} = \lambda_2(\pi_2 - \tau) \tag{25}$$

Using the incentive-compatibility condition $\pi_2 = 2\tau - \pi_1$, we can solve for π_1 as a function of τ:

$$\pi_1 = (a-2)\tau^2 + 2(1+k)\tau - k - \frac{a}{2} \tag{26}$$

Next, conditions (11) and (10) require:

$$\int_\gamma^\tau \{[a\tilde{\gamma} + k - \gamma] - \lambda_2\} \, d\tilde{\gamma} \leq 0 \text{ for } \gamma \in [0, \tau].$$

$$\int_\tau^\gamma \{[a\tilde{\gamma} + k - \gamma] - \lambda_2\} \, d\tilde{\gamma} \geq 0 \text{ for } \gamma \in [\tau, 1].$$

We see that both inequalities are satisfied with equality at $\gamma = \tau$. Hence, we can sign the derivative at τ for each condition, and we get that:

$$-[a\tau + k - \tau] + \lambda_2 \geq 0$$

$$[a\tau + k - \tau] - \lambda_2 \geq 0$$

which implies that

$$\lambda_2 = (a-1)\tau + k \tag{27}$$

Substituting (26) and (27) into (24), it follows that τ must solve the following polynomial equation:

$$2(a-2)^2\tau^3 + 3(a-2)(1+2k)\tau^2 + \left(4k(k+2) - 2a(k-1) - a^2\right)\tau - k(a+2k) = 0 \tag{28}$$

Conditions (26) and (28) correspond to the necessary conditions obtained in Melumad and Shibano (1991) for the optimality of an extreme one-step allocation.

Finally, an incentive-compatible extreme one-step allocation that satisfies (26) and (28) exists if and only if $a > 2$ and $k \in (1-a, 0)$. By construction, this allocation is such that (9) is satisfied for both pooling intervals with $\lambda_2 = (a-1)\tau + k$. Moreover, it is easy to show that $a > 2$ implies that the integral expressions in (11) and (10) are non-decreasing in γ with the previous λ_2. Therefore, any extreme one-step allocation that satisfies (26) and (28) also satisfies Condition 3. We thus obtain Melumad and Shibano (1991)'s characterization of optimal extreme one-step delegation as a special case of Theorem 1.

6.4 A hybrid example

In this section, we show how our theorem can be used to prove the optimality of a hybrid allocation that contains jumps as well as regions of flexibility. For this case, we maintain the distribution from the previous example, that is, $F(\gamma) = \gamma \in [0, 1]$. We also maintain the quadratic preferences for the agent, and let $b(\pi) = -\pi^2/2$. However, we generalize the planner's preferences to be $w(\gamma, \pi) = b(\pi) + h(\gamma)\pi$. In this case, $h(\gamma)$ represents the planner's preferred action for a given γ. This specification lies within the family of cases considered in Alonso and Matouschek (2008).[10]

For this example, we specialize the planner's preferred choice to the following function (symmetric around 1/2):

$$h(\gamma) = \begin{cases} 0 & \text{for } 0 \le \gamma < \frac{1}{2} - \epsilon, \\ \left(\gamma - \frac{1}{2} + \epsilon\right)\frac{1}{2\epsilon} & \text{for } \gamma \in \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right), \\ 1 & \text{for } \frac{1}{2} + \epsilon \le \gamma \le 1, \end{cases}$$

for some value of $\epsilon \in (0, 1/4)$.

We will look for conditions where the solution to the optimal delegation problem takes the form of an exclusion region in the center of the type distribution. Anticipating symmetry, the proposed allocation takes the form:

$$\hat{\pi}(\gamma|a_2) = \begin{cases} \gamma & \text{for } 0 \le \gamma < a_2 \\ a_2 & \text{for } a_2 \le \gamma < 1/2 \\ 1 - a_2 & \text{for } 1/2 \le \gamma < 1 - a_2 \\ \gamma & \text{for } 1 - a_2 \le \gamma < 1 \end{cases} \tag{29}$$

for some value of $a_2 \in (0, 1/2 - \epsilon)$. Note that the above allocation is incentive compatible, as it

[10] Alonso and Matouschek (2008) in Result 6 Section 7 (and their Figure 8) present a very similar example to the one in this section, but where the principal's preferences are linearly increasing over the full support (rather than constant at the extremes as in ours). We choose a different example because it highlights how our methods can be used in non-linear settings. In addition, as we discuss at the end of this subsection, our example can be extended to one where the optimal allocation features a countable number of jumps.

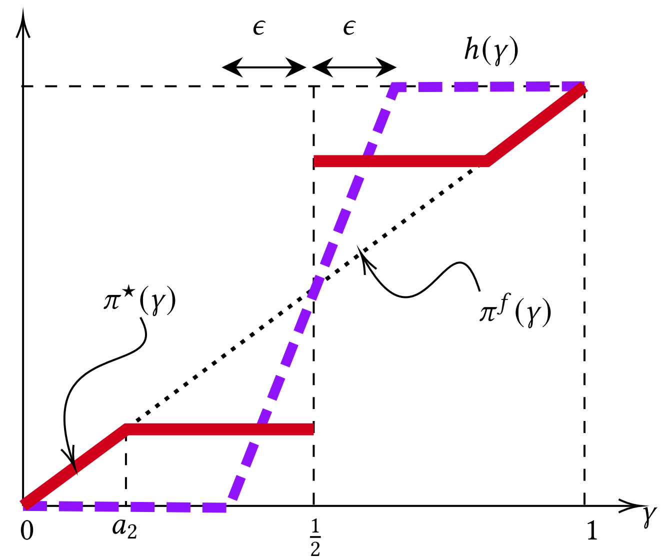


Figure 2: The (symmetric) hybrid example.

is symmetric around the agent's preferred action at the jump at $1/2$. We can plug this allocation into the planner's objective and optimize for a_2. Solving the resulting first-order condition, we obtain that:

$$a_2^\star = \frac{1}{4}\left(1 - \sqrt{1 - 4\epsilon}\right)$$

which is indeed in $(0, 1/2 - \epsilon)$.

The proposed allocation is $\pi^\star(\gamma) = \hat{\pi}(\gamma | a_2^\star)$ for all $\gamma \in \Gamma$. Its associated interval partition is composed of four intervals with boundaries given by $a_1 = 0$, $a_2 = \frac{1}{4}\left(1 - \sqrt{1 - 4\epsilon}\right)$, $a_3 = 1/2$, $a_4 = 1 - a_2$, $a_5 = 1$; with $p_1 = 0$, $p_2 = 1$, $p_3 = 1$, $p_4 = 0$. In addition, $\pi_2 = a_2$ and $\pi_3 = a_4$, while $\gamma_2 = a_2$ and $\gamma_3 = a_4$. Figure 2 displays the proposed allocation, together with the agent's and the principal's preferred choices.

To use Theorem 1, first note that in this case,[11]

$$\kappa(\gamma) = 1, \text{ and } G(\gamma) = F(\gamma) = \gamma.$$

For Condition 3 part (c1), note that

$$G(\gamma) - w_\pi(\gamma, \pi_f(\gamma)) f(\gamma) = 2\gamma - h(\gamma)$$

which, given h, is non-decreasing for $\gamma \in [0, a_2]$ and for $\gamma \in [a_4, 1]$ as required.

Moving towards part (c2) of Condition 3, consider first the region with $p_2 = 1$. Equation (9)

[11] Note that this holds whether we allow for money burning or not.

becomes

$$\int_{a_2}^{a_3} w_\pi(\tilde{\gamma}, \pi_2) f(\tilde{\gamma}) d\tilde{\gamma} = \lambda_3(a_3 - \gamma_2) + \lambda_2(\gamma_2 - a_2).$$

Using that $w_\pi(\tilde{\gamma}, \pi_2) = -a_2 + h(\tilde{\gamma})$, $a_3 = 1/2$, and $\gamma_2 = a_2$, we get

$$\int_{a_2}^{1/2} (-a_2 + h(\tilde{\gamma})) d\tilde{\gamma} = \lambda_3(1/2 - a_2).$$

Using the value of a_2 and h, the integral can be shown to be zero; and thus, given that $a_2 < 1/2$, we have that $\lambda_3 = 0$.

Given that $[a_2, \gamma_2)$ is empty, inequality (10) does not apply. In this case, inequality (11) requires that

$$\int_{\gamma}^{1/2} (-a_2 + h(\tilde{\gamma})) d\tilde{\gamma} \leq (\gamma - a_2)(1/2 - \gamma)$$

for all $\gamma \in (a_2, 1/2]$. Computing the integral for $\gamma \in (a_2, 1/2 - \epsilon]$, the inequality becomes:

$$4\gamma^2 + \epsilon \leq 2\gamma,$$

which holds for $\gamma \in [a_2, 1/2 - \epsilon]$ as required. Computing the integral but now for $\gamma \in (1/2 - \epsilon, 1/2]$, we obtain that

$$-\frac{(1 - 2\gamma)^2(1 - 4\epsilon)}{16\epsilon} \leq 0,$$

which holds. Note that the inequality (12) does not apply as $\gamma_2 = a_2$.

A symmetric argument shows that $\lambda_4 = 0$ and that part (c2) holds for $p_3 = 1$.

The final conditions (c3) and (c4) hold as $w_\pi(\gamma, \pi_f(\gamma)) = 0$ for $\gamma \in \{0, 1\}$.

Taken together, all of the above imply that we can apply Theorem 1 to show that the proposed allocation solves the delegation problem.

An interesting feature of this case is that it can easily be extended to study the optimality of an allocation with multiple jumps. For example, suppose that the support is now $[0, 2]$. Suppose that the agent's preferences remain linear in γ. The preferred choice of the principal is now given by $h_e(\gamma)$ where $h_e(\gamma) = h(\gamma)$ for $\gamma \in [0, 1]$ and $h_e(\gamma) = h(\gamma - 1) + 1$ for $\gamma \in [1, 2]$. Basically, we have "expanded" the principal preferences by repeating the same pattern of $[0, 1]$ over $[1, 2]$ but shifted up so that the principal's preferred choice remains continuous. Consider then the similarly "shifted and expanded" proposed allocation $\pi_e^\star(\gamma)$ where $\pi_e^\star(\gamma) = \pi^\star(\gamma)$ for $\gamma \in [0, 1]$ and $\pi_e^\star(\gamma) = \pi^\star(\gamma - 1) + 1$ for $\gamma \in [1, 2]$. This proposed allocation features two jumps (one at $1/2$ as before and another at $3/2$). We can replicate the steps above for each subinterval $[0, 1]$ and $[1, 2]$ and use Theorem 3 to prove the optimality of the proposed allocation. In this fashion, we can continue to expand the preferences and the preferred allocation, and thus incorporate arbitrarily many jumps into the analysis.

6.5 The hybrid example outside the linear family

We now consider an extension of the hybrid example, where the principal's payoff is given by $w(\pi, \gamma) = A(\gamma)(b(\pi) + h(\gamma)\pi)$, where b and h are as defined in Section 6.4. This example falls outside the linear family in (15) by allowing the relative risk aversion between the principal and the agent to depend on the state. In particular, the relative risk aversion is given by:

$$A(\gamma) = \frac{w_{\pi\pi}(\pi, \gamma)}{b_{\pi\pi}(\pi, \gamma)}$$

Note that if $A(\gamma)$ is constant and equal to 1, we recover the example from the previous section.

The goal of this extension is to demonstrate that our conditions readily extend beyond the family in (15). For this example, we consider the following symmetric expression for $A(\gamma)$.

$$A(\gamma) = \begin{cases} g(\gamma), & \text{for } \gamma \in [0, \alpha] \\ 1, & \text{for } \gamma \in [\alpha, 1-\alpha] \\ g(1-\gamma), & \text{for } \gamma \in [1-\alpha, 1] \end{cases}$$

for some continuous and differentiable function $g(\gamma) \geq 0$ with $g(\alpha) = 1$, and for $\alpha < a_2^\star = \frac{1}{4}\left(1 - \sqrt{1-4\epsilon}\right)$.

No money burning. We are now ready to check Condition 3 in the case without money burning. First note that $\kappa(\gamma) = A(\gamma)$. Note also that $G(a) - G(b) = a - b$ for any $a, b \in [a_2, a_4]$. Thus part (c2) of Condition 3 holds just as before. Now part (c1) requires that

$$\int_0^\gamma A(\tilde{\gamma})d\tilde{\gamma} + A(\gamma)\gamma$$

be non-decreasing for $\gamma \in [0, a_2]$. This is satisfied for $\gamma \in [\alpha, a_2]$. So it suffices that g satisfies:

$$2g(\gamma) + g'(\gamma)\gamma \geq 0; \text{ for } \gamma \in [0, \alpha]. \tag{30}$$

Symmetry implies that (c1) also holds for $\gamma \in [a_4, 1]$. The other conditions of Theorem 1 continue to hold as before, and thus the proposed allocation remains optimal as long as (30) holds.

Money burning. The case with money burning is similar, but now $\kappa(\gamma) = \min\{A(\gamma), 1\}$. We then require that

$$\int_0^\gamma \min\{g(\tilde{\gamma}), 1\} d\tilde{\gamma} + g(\gamma)\gamma$$

be non-decreasing for $\gamma \in [0, \alpha]$. Two simple cases where this holds are when (i) $g(\gamma) \geq 1$ and $(1 + g(\gamma))\gamma$ is non-decreasing, or (ii) $g(\gamma) \leq 1$ and (30) holds.

This example illustrates the more general principle that our framework is robust to certain variations in the principal's risk preferences. Specifically, modifying the principal's risk aversion in separating regions of an optimal allocation does not alter the optimality of that allocation, as long as the monotonicity condition in (c1) is maintained with or without the presence of money burning. This robustness highlights the flexibility and applicability of Condition 3, extending its relevance beyond the family of preferences described by (15).

7 Conclusion

We have provided a sufficient theorem for solving the delegation problem. Its conditions map to known results for interval allocations, while crucially extending them to the cases of discontinuous allocations. We have also provided examples that illustrate the versatility of our approach, both supporting existing understanding and shedding light on new optimal scenarios.

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A Necessary conditions for Optimal Delegation

First, we show that Condition 3 requires that

$$\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} = \kappa(a_{i+1})$$
$$\forall\, \pi \in [\pi_i, \pi_{i+1}] \text{ and } \Gamma_i, \Gamma_{i+1} \text{ s.t. } p_i = p_{i+1} = 1.$$

To see this, let Γ_i, Γ_{i+1} be such that $p_i = p_{i+1} = 1$. This implies $\gamma_i < a_{i+1} < \gamma_{i+1}$. The inequality conditions in (11) for Γ_i and (10) for Γ_{i+1} imply:

$$\int_{\gamma}^{a_{i+1}} [w_\pi(\tilde{\gamma},\pi_i)f(\tilde{\gamma}) - \lambda_{i+1}]\, d\tilde{\gamma} \le (\gamma - \gamma_i)[G(a_{i+1}) - G(\gamma)] \text{ for } \gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}].$$

$$\int_{a_{i+1}}^{\gamma} [w_\pi(\tilde{\gamma},\pi_{i+1})f(\tilde{\gamma}) - \lambda_{i+1}]\, d\tilde{\gamma} \ge (\gamma - \gamma_{i+1})[G(\gamma) - G(a_{i+1})] \text{ for } \gamma \in \Gamma_{i+1} \cap [a_{i+1}, \gamma_{i+1}).$$

Both inequalities are satisfied with equality at $\gamma = a_{i+1}$. Hence, we can sign the derivative at a_{i+1} for each condition , and we get that:

$$-w_\pi(a_{i+1},\pi_i)f(a_{i+1}) + \lambda_{i+1} \ge -(a_{i+1} - \gamma_i)\kappa(a_{i+1})f(a_{i+1})$$
$$w_\pi(a_{i+1},\pi_{i+1})f(a_{i+1}) - \lambda_{i+1} \ge (a_{i+1} - \gamma_{i+1})\kappa(a_{i+1})f(a_{i+1})$$

And rearranging,

$$\begin{aligned} w_\pi(a_{i+1},\pi_i)f(a_{i+1}) - (a_{i+1} - \gamma_i)\kappa(a_{i+1})f(a_{i+1}) &\le \lambda_{i+1} \\ &\le w_\pi(a_{i+1},\pi_{i+1})f(a_{i+1}) - (a_{i+1} - \gamma_{i+1})\kappa(a_{i+1})f(a_{i+1}) \end{aligned} \tag{31}$$

Next, let us define the function

$$m(\pi) \equiv w_\pi(a_{i+1},\pi) - (a_{i+1} + b'(\pi))\kappa(a_{i+1}).$$

Differentiating $m(.)$, we can show that $m(.)$ is non-increasing:

$$m'(\pi) = b''(\pi)\left[\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} - \kappa(a_{i+1})\right] \le 0 \tag{32}$$

where the inequality follows from $b''(\pi) < 0$ and $\kappa(a_{i+1}) \le \frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)}$ for all π. Therefore, $m(\pi_i) \ge$

$m(\pi_{i+1})$. Thus condition (31) can be written as

$$m(\pi_i)f(a_{i+1}) \leq \lambda_{i+1} \leq m(\pi_{i+1})f(a_{i+1})$$

using that $b'(\pi_i) = -\gamma_i$ and $b'(\pi_{i+1}) = -\gamma_{i+1}$. Thus, condition (31) requires that $m(\pi_i) = m(\pi_{i+1})$ given that $f(a_{i+1}) > 0$. Since $m(.)$ is non-increasing, this is equivalent to $m(\pi)$ constant for $\pi \in [\pi_i, \pi_{i+1}]$. It then follows that for Condition 3 to hold, it is required that

$$\frac{w_{\pi\pi}(a_{i+1}, \pi)}{b''(\pi)} = \kappa(a_{i+1}) \text{ for all } \pi \in [\pi_i, \pi_{i+1}].$$

Now, we can show that Condition 3 implies part (ii) of Condition 2. To see this, note that from the above, Condition 3 implies that $\lambda_{i+1} = m(\pi)f(a_{i+1})$ for every $\pi \in [\pi_i, \pi_{i+1}]$. Integrating both sides yields:

$$\lambda_{i+1}(\pi_{i+1} - \pi_i) = \left(\int_{\pi_i}^{\pi_{i+1}} m(\pi)\right) f(a_{i+1}) \tag{33}$$

It suffices to show that $\int_{\pi_i}^{\pi_{i+1}} m(\pi) = w(a_{i+1}, \pi_{i+1}) - w(a_{i+1}, \pi_i)$. Integrating $m(.)$ over the interval $[\pi_i, \pi_{i+1}]$ yields:

$$\int_{\pi_i}^{\pi_{i+1}} m(\pi) = w(a_{i+1}, \pi_{i+1}) - w(a_{i+1}, \pi_i) - \kappa(a_{i+1})(a_{i+1}(\pi_{i+1} - \pi_i) + b(\pi_{i+1}) - b(\pi_i)) \tag{34}$$

Incentive compatibility requires that $a_{i+1}\pi_{i+1} + b(\pi_{i+1}) = a_{i+1}\pi_i + b(\pi_i)$. Therefore, the second term in the right hand side of the equation above is zero, which concludes the proof.

B Proofs

B.1 Auxiliary lemma

The next lemma shows that two incentive-compatible escalators-and-stairs allocations that separate and pool in the same subsets and assign the same action to the lowest type are the same:

Lemma 2. *Let $\boldsymbol{\pi}$ be an incentive-compatible escalators-and-stairs allocation with associated partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$. Let $\hat{\boldsymbol{\pi}}$ be another incentive-compatible escalators-and-stairs allocation with the same associated partition and that separates and pools in the same intervals as $\boldsymbol{\pi}$. If $\boldsymbol{\pi}$ separates in at least one subset Γ_i, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$. Otherwise, if $\boldsymbol{\pi}(\underline{\gamma}) = \hat{\boldsymbol{\pi}}(\underline{\gamma})$, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$.*

Proof. Suppose that there exists i such that for all $\gamma \in \Gamma_i$, $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$. Then $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$ for all $\gamma \in \Gamma'$ for any adjacent subset Γ' to Γ_i. To see this note first that if Γ' separates, then $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma) = \pi^f(\gamma)$ for all $\gamma \in \Gamma'$. If Γ' pools and Γ_i also pools, then Condition 1 part (i) uniquely determines the pooling action at Γ'. Finally, if Γ_i separates and Γ' pools, then Condition 1 part (ii) uniquely determines the pooling action at Γ'. By induction, we can then show that if there exists i such that for all $\gamma \in \Gamma_i$, $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$.

Now, to prove the lemma, if there is a subset where $\boldsymbol{\pi}$ separates, by construction, both allocations prescribe the same action in that subset and thus $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$. If there is no subset where $\boldsymbol{\pi}$ separates, it suffices that the prescribed actions for the lowest type are equal in both allocations.

B.2 Proof of Proposition 1

Proof. Using that $\boldsymbol{t}(\gamma) = 0$ for all $\gamma \in \Gamma$, from Milgrom and Segal (2002), we have that incentive compatibility is equivalent to monotonicity of the allocation and the following integral condition:

$$\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) = \int_{\gamma_0}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma} + \gamma_0\boldsymbol{\pi}(\gamma_0) + b(\boldsymbol{\pi}(\gamma_0))$$

for all $\gamma \in \Gamma$ and any baseline $\gamma_0 \in \Gamma$.

Consider first the necessity of conditions (i) and (ii) in Condition 1.

Necessity of condition (i). Consider $\gamma_0 \in \Gamma_i$ and $\gamma' \in \Gamma_{i+1}$. Then incentive compatibility requires that $\pi_{i+1} \geq \pi_i$. Given that we impose that no two adjacent intervals pool at the same action, it follows that $\pi_{i+1} > \pi_i$. The integral form of the incentive constraint then requires

$$\gamma'\pi_{i+1} + b(\pi_{i+1}) = \int_{\gamma_0}^{\gamma} \pi_i d\tilde{\gamma} + \int_{\gamma}^{\gamma'} \pi_{i+1} d\tilde{\gamma} + \gamma_0\pi_i + b(\pi_i),$$

for $\gamma = a_{i+1}$; which delivers

$$\gamma\pi_{i+1} + b(\pi_{i+1}) = \gamma\pi_i + b(\pi_i).$$

Necessity of condition (ii). Without loss of generality assume that Γ_i separates, and Γ_{i+1} pools at x. Let $\gamma_0 \in \Gamma_i$ and $\gamma' \in \Gamma_{i+1}$. Then

$$\gamma' x + b(x) = \int_{\gamma_0}^{\gamma} \pi^f(\tilde{\gamma})d\tilde{\gamma} + \int_{\gamma}^{\gamma'} x d\tilde{\gamma} + \gamma_0\pi^f(\gamma_0) + b(\pi^f(\gamma_0)),$$

for $\gamma = a_{i+1}$. Taking the limit of the above as $\gamma_0 \to \gamma$, and using the continuity of the flexible allocation function π^f, we have

$$\gamma' x + b(x) = \int_{\gamma}^{\gamma'} x d\tilde{\gamma} + \gamma \pi^f(\gamma) + b(\pi^f(\gamma)),$$

which implies $\gamma x + b(x) = \gamma \pi^f(\gamma) + b(\pi^f(\gamma))$. Uniqueness of the flexible allocation implies that $x = \pi^f(\gamma)$.

Sufficiency. Note that an allocation that satisfies conditions (i) and (ii) is monotone. Given that in the areas where the allocation separates, the flexible allocation is offered to the corresponding types, the only concern for incentive compatibility arises for types in pooling regions.

Consider a type γ in a region Γ_i that pools at x^o. Let $\tilde{\gamma} = \sup \Gamma_i = a_{i+1}$. Define x' to be either $x' = x^o = \pi^f(\tilde{\gamma})$ if Γ_{i+1} separates or $x' \neq x^o$ and given by:

$$\tilde{\gamma} x' + b(x') = \tilde{\gamma} x^o + b(x^o),$$

if Γ_{i+1} pools at x'. Note that in this latter case, $x' > \pi^f(\tilde{\gamma}) > x^o$.

Type $\tilde{\gamma}$ then either prefers the choice of x^o to any alternative (because it is its flexible choice) or is indifferent between x^o and a higher alternative x'. Given that for any type $\gamma' \geq \tilde{\gamma}$ we have that $\boldsymbol{\pi}(\gamma') \geq x'$ (by monotonicity), it follows that type γ must prefer x^o to *any* other action prescribed to any higher type. A similar argument shows that γ also prefers its prescribe action to that of any lower type.

B.3 Proof of Lemma 1

Proof. Given $p = \{p_i\}_{i=1}^n \in P_n$, let $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ be a solution to Problem 1. Define ϵ to be such that

$$0 < \epsilon < \min_{i \in \{1,\ldots,n\}} \{a_{i+1}^\star - a_i^\star\}$$

Such a value ϵ exists given that $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ satisfies (??) given p. Now, consider the following version of Problem 1:

$$\max_{\{a_i,\pi_i\}_{i=1}^n} \sum_{i=1}^n \left(p_i \int_{a_i}^{a_{i+1}} w(\gamma,\pi_i)f(\gamma)d\gamma + (1-p_i)\int_{a_i}^{a_{i+1}} w(\gamma,\pi_f(\gamma))f(\gamma)d\gamma\right) \tag{35}$$

subject to:

$$a_{i+1}\pi_i + b(\pi_i) = a_{i+1}\pi_{i+1} + b(\pi_{i+1}), \qquad \text{if } p_i = p_{i+1} = 1;$$

$$\pi_i = \pi_f(a_{i+1}), \qquad \text{if } p_i = 1, p_{i+1} = 0;$$

$$\pi_f(a_{i+1}) = \pi_{i+1}, \qquad \text{if } p_i = 0, p_{i+1} = 1;$$

$$\pi_{i+1} \geq \pi_i, a_{i+1} \geq a_i + \epsilon, \pi_i \in \Pi, a_i \in \Gamma, \forall i \in \{1,\ldots,n\}.$$

where we have replaced the constraint $a_{i+1} > a_i$ with the stricter (but closed) constraint $a_{i+1} \geq a_i + \epsilon$. If $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ solves Problem 1, then it must also solve this new Problem 35 as every allocation in the constraint set of Problem 35 is also in 1. We will use this new problem to derive necessary conditions.

Let i be such that $p_i = 1$. We consider the following cases.

First, consider the case that $p_{i+1} = 0$ for $i \leq n-1$. From the constraints, we then have that $\pi_i^\star = \pi_f(a_{i+1}^\star)$. Moreover, note that, if $i \geq 2$, then $p_{i-1} = 1$ (as $p \in P_n$). Setting $\pi_i^\star = \pi_f(a_{i+1}^\star)$ in the constraint corresponding to $p_{i-1} = p_i = 1$ yields that $a_i^\star \pi_{i-1}^\star + b(\pi_{i-1}^\star) = a_i^\star \pi_f(a_{i+1}^\star) + b(\pi_f(a_{i+1}^\star))$. Let λ_i be the Lagrange multiplier associated with this constraint (if $i = 1$, then simply set $\lambda_1 = 0$). Moreover, we plug $\pi_i^\star = \pi_f(a_{i+1}^\star)$ into the objective function. Note that $a_{i+1}^\star$ shows up in the expression for the objective function only in the following term:

$$\int_{a_i^\star}^{a_{i+1}^\star} w(\gamma,\pi_f(a_{i+1}^\star))f(\gamma)d\gamma + \int_{a_{i+1}^\star}^{a_{i+2}^\star} w(\gamma,\pi_f(\gamma))f(\gamma)d\gamma$$

The FOC with respect to a_{i+1} then yields the condition:

$$\left(\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma,\pi_f(a_{i+1}^\star))f(\gamma)d\gamma\right)\pi_f'(a_{i+1}^\star) = -\lambda_i\left[a_i^\star + b'(\pi_f(a_{i+1}^\star)\right]\pi_f'(a_{i+1}^\star)$$

Using that $\pi_f(a_{i+1}^\star) = \pi_i^\star$, $\pi_f'(a_{i+1}^\star) > 0$ and $b'(\pi_f(a_{i+1}^\star)) = -a_{i+1}^\star$, gives us:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma,\pi_i^\star)f(\gamma)d\gamma = -\lambda_i(a_i^\star - a_{i+1}^\star) \tag{36}$$

Finally, from the definition of γ_i, it follows that $\gamma_i = a_{i+1}$. We can thus write the above equality as:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma, \pi_i^\star) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1}^\star - \gamma_i) - \lambda_i(a_i^\star - \gamma_i),$$

for an arbitrary λ_{i+1} (as $\gamma_i = a_{i+1}^\star$, the value of λ_{i+1} is irrelevant). This implies that condition (i) is necessary.

For $p_{i-1} = 0$ and $i \geq 2$ a similar argument delivers the proof for the necessity of condition (i), so we omit it.

Next, assume that $p_i = 1$ and there is no adjacent separating interval. Then it must be that $p_{i-1} = 1$ for $i \geq 2$ and $p_{i+1} = 1$ for $i \leq n-1$. From the constraints, we then have that $a_i^\star \pi_{i-1}^\star + b(\pi_{i-1}^\star) = a_i^\star \pi_i^\star + b(\pi_i^\star)$ for $i \geq 2$ and $a_{i+1}^\star \pi_i^\star + b(\pi_i^\star) = a_{i+1}^\star \pi_{i+1}^\star + b(\pi_{i+1}^\star)$ for $i \leq n-1$. Let λ_i be the Lagrange multiplier associated with the first constraint (setting $\lambda_i = 0$ if $i = 1$) and λ_{i+1} be the Lagrange multiplier associated with the second constraint (setting $\lambda_{n+1} = 0$ if $i = n$). Note that $\pi_i^\star$ shows up in both constraints, while a_{i+1}^* appers in the latter. Moreover, $\pi_i^\star$ and a_{i+1}^* also appear in the following terms in the objective function:

$$\int_{a_i^\star}^{a_{i+1}^\star} w(\gamma, \pi_i^\star) f(\gamma) d\gamma + \int_{a_{i+1}^\star}^{a_{i+2}^\star} w(\gamma, \pi_{i+1}^\star) f(\gamma) d\gamma$$

Taking the FOC with respect to $\pi_i^\star$ yields the following condition:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma, \pi_i^\star) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1}^\star + b'(\pi_i^\star)) - \lambda_i(a_i^\star + b'(\pi_i^\star)) \tag{37}$$

Using that $\gamma_i = -b'(\pi_i^\star)$ implies that (i) is necessary. Taking the FOC with respect to $a_{i+i}^\star$ yields:

$$(w(a_{i+1}^*, \pi_i^\star) - w(a_{i+1}^*, \pi_{i+1}^\star)) f(a_{i+1}) = \lambda_{i+1}(\pi_i^\star - \pi_{i+1}^\star) \tag{38}$$

This shows that (ii) is necessary, which concludes the proof. □

B.4 Proof of Theorem 1

Proof. We prove each part of the theorem separately. For both cases, let $\boldsymbol{\pi}^* : \Gamma \to \Pi$ denote the proposed optimal allocation with interval partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ such that Condition 3 is satisfied.

Part (a)

Consider the problem without money-burning, i.e. Problem 1 with constraint (2). By writing the incentive constraints in their usual integral form plus a monotonicity restriction, we can rewrite the problem as:

$$\max_{\{\boldsymbol{\pi}:\Gamma\to\Pi\}} \int_{\Gamma} w(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) d\gamma \quad \text{subject to:} \tag{39}$$

$$\gamma \boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) = \int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma}) d\tilde{\gamma} + \underline{U} \quad \forall \gamma \in \Gamma \tag{40}$$

$$\boldsymbol{\pi} \text{ non-decreasing} \tag{41}$$

where $\underline{U} = \underline{\gamma}\boldsymbol{\pi}(\underline{\gamma}) + b(\boldsymbol{\pi}(\underline{\gamma}))$.

Rewriting the IC constraints. We first embed the monotonicity constraint (41) into the choice set of $\boldsymbol{\pi}$. We then, write constraint (40) as two inequalities:

$$\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma}) d\tilde{\gamma} + \underline{U} - \gamma \boldsymbol{\pi}(\gamma) - b(\boldsymbol{\pi}(\gamma)) \leq 0 \quad \forall \gamma \in \Gamma \tag{42}$$

$$-\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma}) d\tilde{\gamma} - \underline{U} + \gamma \boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) \leq 0 \quad \forall \gamma \in \Gamma \tag{43}$$

A Lagrangian. By assigning cumulative Lagrange multiplier functions Λ_1 and Λ_2 to constraints (42) and (43), respectively, we can write the Lagrangian for the problem:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi} \mid \Lambda_1, \Lambda_2) \equiv & \int_{\Gamma} w(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) d\gamma \\ & - \int_{\Gamma} \left(\int_{\underline{\gamma}}^{\gamma} \pi(\gamma') d\gamma' + \underline{U} - \gamma \pi(\gamma) - b(\boldsymbol{\pi}(\gamma)) \right) d\left(\Lambda_1(\gamma) - \Lambda_2(\gamma)\right).\end{aligned}$$

The Lagrange multipliers Λ_1 and Λ_2 are restricted to be non-decreasing functions. Let $\Lambda(\gamma) \equiv \Lambda_1(\gamma) - \Lambda_2(\gamma)$. Integrating by parts the Lagrangian, we get:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi} \mid \Lambda) = & \int_{\Gamma} [w(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) - (\Lambda(\overline{\gamma}) - \Lambda(\gamma)) \boldsymbol{\pi}(\gamma)] d\gamma \\ & + \int_{\Gamma} (\gamma \pi(\gamma) + b(\boldsymbol{\pi}(\gamma))) d\Lambda(\gamma) - \underline{U}(\Lambda(\overline{\gamma}) - \Lambda(\underline{\gamma}))\end{aligned}$$

Our objective is to explicitly construct valid Lagrange multipliers, $\Lambda_1(\gamma)$ and $\Lambda_2(\gamma)$, such that the resulting Lagrangian is concave in $\boldsymbol{\pi}$ when evaluated at the multiplier and the first-order conditions are satisfied at the proposed allocation $\boldsymbol{\pi}^*$. Amador and Bagwell (2013) show that, under these conditions, $\boldsymbol{\pi}^*$ is optimal among all IC allocations.

A proposed multiplier. Let us propose some non-decreasing multipliers Λ_1 and Λ_2 so that their difference, Λ, satisfies $\Lambda(\underline{\gamma}) = \Lambda(\overline{\gamma}) = 0$ and the following conditions for all $\gamma \in (\underline{\gamma}, \overline{\gamma})$:

$$\Lambda(\gamma) = \begin{cases} -w_\pi(\gamma, \pi_f(\gamma))f(\gamma) & p_i = 0, \gamma \in \Gamma_i \\ -\lambda_i + G(a_i) - G(\gamma) & p_i = 1, \gamma \in [a_i, \gamma_i) \cap \Gamma_i \\ -\lambda_{i+1} + G(a_{i+1}) - G(\gamma) & p_i = 1, \gamma \in [\gamma_i, a_{i+1}) \cap \Gamma_i \end{cases}$$

Checking that the multipliers are non-decreasing. We now show that Condition 3 guarantees that $G(\gamma) + \Lambda(\gamma) \equiv R(\gamma)$ is non-decreasing, which then implies that $\Lambda(\gamma)$ can be written as the difference of two non decreasing functions, $R(\gamma) - G(\gamma)$.

First, we show that $G(\gamma) + \Lambda(\gamma)$ is non-decreasing in the interior of each interval Γ_i. Take $i \in \{1, \dots, n\}$. If $p_i = 0$, then this follows immediately from (c1) which states that $G(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ is non decreasing for all $\gamma \in \Gamma_i$. If $p_i = 1$, we consider two cases. If $a_i < \gamma_i < a_{i+1}$, then $G(\gamma) + \Lambda(\gamma)$ is piecewise constant with one jump at γ_i. Condition (12) from (c2) immediately implies that the jump at γ_i is non-negative. Otherwise, $G(\gamma) + \Lambda(\gamma)$ is constant for all $\gamma \in \Gamma_i$.

Next, let us show that the jumps at $\{a_i\}_{i=1}^{n+1}$ are non-negative.

- Take $i = 1$. We consider the following two cases.

  First, let $p_1 = 0$. Then it immediately follows from (c3) that the jump at $a_1 = \underline{\gamma}$:

  $$-w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) - \Lambda(\underline{\gamma})$$

  is non-negative.

  Next, consider the case where $p_1 = 1$.

  - If $a_1 < \gamma_1$, then note that $\lambda_1 = 0$ implies that $\Lambda(\gamma)$ is continuous at $a_1 = \underline{\gamma}$ and therefore there is no jump (hence, it is non-negative).
  - If $a_1 > \gamma_1$, then suffices to show that $G(a_2) - \lambda_2 \geq \Lambda(\underline{\gamma}) = 0$. Evaluating (11) at $\gamma = a_1$ and using the equality condition (9) to replace the integral term, we have that (c2) implies $\lambda_2(a_1 - \gamma_1) + \lambda_1(\gamma_1 - a_1) \leq [a_1 - \gamma_1]G(a_2)$. The inequality $G(a_2) - \lambda_2 \geq 0$ then follows from the fact that $\lambda_1 = 0$ and $a_1 > \gamma_1$, which shows that the jump at a_1 is non-negative.

- Finally, consider the case $a_1 = \gamma_1$. In this case, the equality condition (9) implies that (11) holds with equality as γ converges to a_1 from the right, i.e. $\gamma \to a_1^+$. Hence, we can sign the derivative as $\gamma \to a_1^+$, and we get that $-w_\pi(a_1, \pi_1) f(a_1) + \lambda_2 \leq G(a_2)$. From the definition of γ_1, we have that $\gamma_1 = a_1$ implies $\pi_1 = \pi_f(a_1)$. It then follows from condition (c3) and $a_1 = \underline{\gamma}$ that $G(a_2) - \lambda_2 \geq -w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) f(\underline{\gamma}) \geq 0$, which concludes the proof for $i = 1$.

- For $i = n$: a similar argument using (9) and (10) from (c2), (c4) and $\lambda_{n+1} = 0$ shows that the jump at $a_{n+1} = \overline{\gamma}$ is also non-negative.

- Finally, let $1 < i < n$. We show that the jumps at a_i and a_{i+1} are non-negative.

  - If $p_i = 0$, then the jumps are:

$$-\lambda_{i-1} + G(a_{i-1}) - G(a_i) \leq -w_\pi(a_i, \pi_f(a_i)) f(a_i) \tag{44}$$

$$-\lambda_{i+2} + G(a_{i+2}) - G(a_{i+1}) \geq -w_\pi(a_{i+1}, \pi_f(a_{i+1})) f(a_{i+1}) \tag{45}$$

    Recall that $p_{i-1} = 1$ and $\gamma_{i-1} = a_i$. Consider condition (10) from (c2) for interval $i - 1$:

$$\int_{a_{i-1}}^{\gamma} \left[w_\pi(\tilde{\gamma}, \pi_{i-1}) f(\tilde{\gamma}) - \lambda_{i-1} \right] d\tilde{\gamma} \geq (\gamma - a_i) \left[G(\gamma) - G(a_{i-1}) \right], \; \gamma \in [a_{i-1}, a_i) \tag{46}$$

    It is immediate that the RHS converges to 0 as $\gamma \to a_i^-$. From equation (9), this must also be the case for the LHS. Hence, (10) holds with equality as $\gamma \to a_i^-$. We can then sign the derivative at $\gamma = a_i$ which yields $w_\pi(a_i, \pi_{i-1}) f(a_i) - \lambda_{i-1} \leq G(a_i) - G(a_{i-1})$. Rearranging terms and using the fact that $\pi_{i-1} = \pi_f(a_i)$ yields $-\lambda_{i-1} + G(a_{i-1}) - G(a_i) \leq -w_\pi(a_i, \pi_f(a_i)) f(a_i)$, which shows that the first jump is non-negative. A similar argument using (9) and (11) for interval $i + 1$ shows that the the second jump is also non-negative.

  - Now, consider $p_i = 1$. If $p_{i+1} = 0$, then by our previous argument, the jump at a_{i+1} is non-negative. If $p_{i+1} = 1$, then note that $\Lambda(\gamma)$ is continuous at a_{i+1} and therefore there is no jump (hence, it is non-negative). The same arguments hold for the jump at a_i.

Concavity of the Lagrangian. Following Amador and Bagwell (2013), we can write the Lagrangian as:

$$\mathcal{L}(\boldsymbol{\pi} \mid \Lambda) = \int_\Gamma [w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma)(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma) d\gamma + \int_\Gamma \Lambda(\gamma)\boldsymbol{\pi}(\gamma) d\gamma + \int_\Gamma (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma))) d(G(\gamma) + \Lambda(\gamma))$$

By the definition of $\kappa(\gamma)$ in equation (6), we see that $w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma)b(\boldsymbol{\pi}(\gamma))$ is concave in $\boldsymbol{\pi}(\gamma)$; further, we just showed that Condition 3 implies that $G(\gamma) + \Lambda(\gamma)$ is non-decreasing. Hence, the above Lagrangian is concave at the proposed multiplier.

Maximizing the Lagrangian. We now proceed to show that the proposed allocation $\boldsymbol{\pi}^*$ maximizes the Lagrangian.

For our problem, taking the Gateaux differential in direction $\boldsymbol{x} \in \hat{\Phi}$ and using that $b'(\pi_f(\gamma)) = -\gamma$ and $b'(\pi_i) = -\gamma_i$ we get that:

$$\partial\mathcal{L}(\boldsymbol{\pi}^*|\Lambda) = \int_\Gamma [w_\pi(\gamma, \boldsymbol{\pi}^*(\gamma)) f(\gamma) + \Lambda(\gamma)] \boldsymbol{x}(\gamma) d\gamma + \sum_{\{i|p_i=1\}} \int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] \boldsymbol{x}(\gamma) d\Lambda(\gamma)$$

which can be rewritten as:

$$\partial\mathcal{L}(\boldsymbol{\pi}^*|\Lambda) = \sum_{\{i|p_i=1\}} \left\{ \int_{a_i}^{a_{i+1}} [w_\pi(\gamma, \pi_i) f(\gamma) + \Lambda(\gamma)] \boldsymbol{x}(\gamma) d\gamma + \int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] \boldsymbol{x}(\gamma) d\Lambda(\gamma) \right\} \quad (47)$$

Let us work with each interval separately. Integrating by parts over Γ_i yields:

$$\left[\int_{a_i}^{a_{i+1}} [w_\pi(\gamma, \pi_i) f(\gamma) + \Lambda(\gamma)] d\gamma\right] \boldsymbol{x}(a_{i+1}) - \int_{a_i}^{a_{i+1}} \left[\int_{a_i}^{\gamma} [w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) + \Lambda(\tilde{\gamma})] d\tilde{\gamma}\right] d\boldsymbol{x}(\gamma) + \left[\int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] d\Lambda(\gamma)\right] \boldsymbol{x}(a_{i+1}) - \int_{a_i}^{a_{i+1}} \left[\int_{a_i}^{\gamma} [\tilde{\gamma} - \gamma_i] d\Lambda(\tilde{\gamma})\right] d\boldsymbol{x}(\gamma)$$

We require that the sum over the terms above be nonpositive for all nondecreasing $\boldsymbol{x}$ and zero when evaluated at $\boldsymbol{x} = \boldsymbol{\pi}^*$. Note that for $\gamma \in \Gamma_i$ such that $p_i = 1$, if $\boldsymbol{x} = \boldsymbol{\pi}^*$, then $d\boldsymbol{x}(\gamma) = 0$. So we need that:

$$\int_{a_i}^{\gamma}\left[w_\pi\left(\tilde{\gamma},\pi_i\right)f(\tilde{\gamma})+\Lambda(\tilde{\gamma})\right]d\tilde{\gamma}+\int_{a_i}^{\gamma}[\tilde{\gamma}-\gamma_i]d\Lambda(\tilde{\gamma})\geq 0 \tag{48}$$
$$\forall\,\Gamma_i \text{ s.t. } p_i=1, \forall\gamma\in\Gamma_i, \text{with equality for } \gamma=a_{i+1}.$$

We can simplify this optimality condition by integrating by parts the second term. We can then state it in the following way,

$$\int_{a_i}^{\gamma} w_\pi\left(\tilde{\gamma},\pi_i\right)f(\tilde{\gamma})d\tilde{\gamma}\geq -\Lambda(\gamma)(\gamma-\gamma_i)+\Lambda(a_i)(a_i-\gamma_i) \tag{49}$$
$$\forall\,\Gamma_i \text{ s.t. } p_i=1, \forall\gamma\in\Gamma_i, \text{with equality for } \gamma=a_{i+1}.$$

First, we show that the equality constraint corresponds to the equality condition in (9). It suffices to show that $\Lambda(a_i)=-\lambda_i$ if $\gamma_i\neq a_i$ and $\Lambda(a_{i+1})=-\lambda_{i+1}$ if $\gamma_i\neq a_{i+1}$. If $i=1$, then $\Lambda(a_1)=0=-\lambda_1$. If $i>1$, then $a_i<\gamma_i$.[12] This, in turn, implies that $\Lambda(a_i)=-\lambda_i$. Likewise, if $i=n$, then $\Lambda(a_{n+1})=0=-\lambda_{n+1}$. Otherwise, $\gamma_i<a_{i+1}$, which then implies that $\Lambda(a_{i+1})=-\lambda_{i+1}$.

Moreover, consider (49) for the case where $\gamma\in[a_i,\gamma_i)\cap\Gamma_i$. Replacing the expression for $\Lambda(\gamma)$ yields:

$$\int_{a_i}^{\gamma} w_\pi\left(\tilde{\gamma},\pi_i\right)f(\tilde{\gamma})d\tilde{\gamma}\geq -(-\lambda_i+G(a_i)-G(\gamma))(\gamma-\gamma_i)-\lambda_i(a_i-\gamma_i) \tag{50}$$

We can then rewrite the above expression to get:

$$\int_{a_i}^{\gamma}[w_\pi\left(\tilde{\gamma},\pi_i\right)f(\tilde{\gamma})-\lambda_i]d\tilde{\gamma}\geq(\gamma-\gamma_i)[G(\gamma)-G(a_i)] \tag{51}$$

which corresponds to the inequality condition in (10).

Finally, consider (49) for the case where $\gamma\in[\gamma_i,a_{i+1})\cap\Gamma_i$. We can use (9) to rewrite the inequality as:

$$\int_{\gamma}^{a_{i+1}} w_\pi\left(\tilde{\gamma},\pi_i\right)f(\tilde{\gamma})d\tilde{\gamma}-\lambda_{i+1}(a_{i+1}-\gamma_i)\leq\Lambda(\gamma)(\gamma-\gamma_i)-(\Lambda(a_i)+\lambda_i)(a_i-\gamma_i) \tag{52}$$

Note that the last term in the RHS is zero since $\Lambda(a_i)=-\lambda_i$ if $a_i\neq\gamma_i$. We can once again

[12] Recall that $\gamma_i=-b'(\pi_i)$. Hence, $a_i>\gamma_i$, implies that $\pi^f(a_i)>\pi_i$. Incentive compatibility then implies that $i=1$. A similar argument shows that $a_{i+1}<\gamma_i$ can only occur if $i=n$.

replace the expression for $\Lambda(\gamma)$ to get:

$$\int_{\gamma}^{a_{i+1}} w_{\pi}\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_{i+1}(a_{i+1} - \gamma_i) \leq (-\lambda_{i+1} + G(a_{i+1}) - G(\gamma))(\gamma - \gamma_i) \tag{53}$$

We can then rewrite the above expression to get:

$$\int_{\gamma}^{a_{i+1}} \left[w_{\pi}(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) - \lambda_{i+1}\right] d\tilde{\gamma} \leq (\gamma - \gamma_i)\left[G(a_{i+1}) - G(\gamma)\right], \tag{54}$$

which corresponds to the inequality condition in (11). □

Part (b)

Consider the problem with money-burning. Using the integral form for the incentive constraints, Problem 1 becomes:

$$\max_{\{\boldsymbol{\pi}, \boldsymbol{t}\}} \int_{\Gamma} [w(\gamma, \boldsymbol{\pi}(\gamma)) - \boldsymbol{t}(\gamma)] f(\gamma) d\gamma \quad \text{subject to:} \tag{55}$$

$$\gamma \boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) - \boldsymbol{t}(\gamma) = \int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma}) d\tilde{\gamma} + \underline{U} \quad \forall \gamma \in \Gamma \tag{56}$$

$$\boldsymbol{\pi} \text{ non-decreasing} \tag{57}$$

$$\boldsymbol{t}(\gamma) \geq 0 \quad \forall \gamma \in \Gamma \tag{58}$$

where $\underline{U} = \underline{\gamma} \boldsymbol{\pi}(\underline{\gamma}) + b(\boldsymbol{\pi}(\underline{\gamma})) - \boldsymbol{t}(\underline{\gamma})$.

Rewriting the IC and no-money burning constraints. Solving the integral equation for the money-burning term $\boldsymbol{t}(\gamma)$ and substituting it into the objective function and the non-negativity constraint, yields the following equivalent problem:

$$\max_{\{\boldsymbol{\pi}: \Gamma \to \Pi, \, \boldsymbol{t}(\underline{\gamma}) \geq 0\}} \int_{\Gamma} v(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) + (1 - F(\gamma)) \boldsymbol{\pi}(\gamma)) d\gamma + \underline{U} \quad \text{subject to:} \tag{59}$$

$$\gamma \boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) - \int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma}) d\tilde{\gamma} - \underline{U} \geq 0 \tag{60}$$

$$\boldsymbol{\pi} \text{ non-decreasing} \tag{61}$$

where we define $v(\gamma, \boldsymbol{\pi}(\gamma)) \equiv w(\gamma, \boldsymbol{\pi}(\gamma)) - \gamma\boldsymbol{\pi}(\gamma) - b(\boldsymbol{\pi}(\gamma))$. Note that the money burning term $\boldsymbol{t}(\gamma)$ can be recovered once we solve for $\boldsymbol{\pi}(\gamma)$ and $\boldsymbol{t}(\underline{\gamma})$ via:

$$\boldsymbol{t}(\gamma) = \gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) - \int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma} - \underline{U} \tag{62}$$

A Lagrangian. By assigning a cumulative Lagrange multiplier function $\tilde{\Lambda}$ to constraint (60) we can write the Lagrangian for the problem as:

$$\begin{aligned}\mathcal{L}\left(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \tilde{\Lambda}\right) \equiv & \int_{\Gamma} v(\gamma, \boldsymbol{\pi}(\gamma))f(\gamma) + (1 - F(\gamma))\boldsymbol{\pi}(\gamma))d\gamma + \underline{U} \\ & - \int_{\Gamma}\left(\int_{\underline{\gamma}}^{\gamma} \pi(\gamma')\, d\gamma' + \underline{U} - \gamma\pi(\gamma) - b(\boldsymbol{\pi}(\gamma))\right) d\tilde{\Lambda}(\gamma).\end{aligned}$$

It is required that the Lagrange multiplier $\tilde{\Lambda}$ be nondecreasing.

Integrating by parts the Lagrangian and setting $\tilde{\Lambda}(\bar{\gamma}) = 1$ without loss of generality, we get:

$$\begin{aligned}\mathcal{L}\left(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \tilde{\Lambda}\right) = & \int_{\Gamma}[v(\gamma, \boldsymbol{\pi}(\gamma))f(\gamma) + (\tilde{\Lambda}(\gamma) - F(\gamma))\boldsymbol{\pi}(\gamma)]d\gamma \\ & + \int_{\Gamma}(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))d\tilde{\Lambda}(\gamma) + \underline{U}\tilde{\Lambda}(\underline{\gamma})\end{aligned}$$

A proposed multiplier. Our proposed multiplier in this case is $\tilde{\Lambda}(\gamma) = \Lambda(\gamma) + F(\gamma)$, where $\Lambda(\gamma)$ is such that $\Lambda(\underline{\gamma}) = \Lambda(\bar{\gamma}) = 0$ and:

$$\Lambda(\gamma) = \begin{cases} -w_{\pi}(\gamma, \pi_f(\gamma))f(\gamma) & p_i = 0, \gamma \in \Gamma_i \\ -\lambda_i + G(a_i) - G(\gamma) & p_i = 1, \gamma \in [a_i, \gamma_i) \cap \Gamma_i \\ -\lambda_{i+1} + G(a_{i+1}) - G(\gamma) & p_i = 1, \gamma \in [\gamma_i, a_{i+1}) \cap \Gamma_i \end{cases}$$

Checking the multiplier is non-decreasing. Note that $\Lambda(\gamma)$ is the proposed multiplier for Part (a). Let us show that Condition 3 guarantees that $\tilde{\Lambda}(\gamma)$ is non-decreasing. We write the multiplier as $\tilde{\Lambda}(\gamma) = (\Lambda(\gamma) + G(\gamma)) + (F(\gamma) - G(\gamma))$. We showed in Part (a) that Condition 3 implies $\Lambda(\gamma) + G(\gamma)$ non-decreasing for any given $\kappa(\gamma)$. Therefore, it suffices to show that $F(\gamma) - G(\gamma)$ is non-decreasing with $\kappa(\gamma)$ given by (7). From the definition of F and G, we can write the difference as:

$$\int_{\underline{\gamma}}^{\gamma} (1 - \kappa(\tilde{\gamma}))f(\tilde{\gamma})d(\tilde{\gamma}) \tag{63}$$

Since $\kappa(\gamma) \leq 1$ for every $\gamma \in \Gamma$, it follows that $F(\gamma) - G(\gamma)$ is non-decreasing in γ.

Concavity of the Lagrangian. Let us check that the Lagrangian is concave in the allocation at the proposed multiplier. We can rewrite the Lagrangian as:

$$\mathcal{L}(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \Lambda) = \int_\Gamma [w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma)(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma) d\gamma \\ + \int_\Gamma (\tilde{\Lambda}(\gamma) - F(\gamma))\boldsymbol{\pi}(\gamma) d\gamma + \int_\Gamma (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma))) d(G(\gamma) - F(\gamma) + \tilde{\Lambda}(\gamma))$$

where we have used the fact that $\tilde{\Lambda}(\underline{\gamma}) = 0$, which follows from $\Lambda(\underline{\gamma}) = 0$. The term $w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma)b(\boldsymbol{\pi}(\gamma))$ is concave given the definition of $\kappa(\gamma)$. Finally, note that $G(\gamma) - F(\gamma) + \tilde{\Lambda}(\gamma) = \Lambda(\gamma) + G(\gamma)$, which is non-decreasing given Condition 3. This is needed in the second integral to guarantee that the concavity of b is not reversed.

Maximizing the Lagrangian. That the Lagrangian is maximized at the proposed allocation is similar to the argument used in our proof for Part (a) of Theorem 1 (the no money burning case). To see this, first note that $\boldsymbol{t}(\underline{\gamma})$ does not appear in the Lagrangian, given the proposed Lagrange multiplier. This implies that we can restrict attention to maximizing the Lagrangian over just $\boldsymbol{\pi}(\gamma)$. Moreover, using the fact that $\Lambda(\gamma) = \tilde{\Lambda}(\gamma) - F(\gamma)$, we can rewrite the Lagrangian as:

$$\mathcal{L}(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \Lambda) = \int_\Gamma [w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma)(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma) d\gamma \\ + \int_\Gamma \Lambda(\gamma)\boldsymbol{\pi}(\gamma) d\gamma + \int_\Gamma (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma))) d(G(\gamma) + \Lambda(\gamma))$$

which is equivalent to the Lagrangian in the proof of Part (a) of Theorem 1 with κ given by (7), and where Λ is the Lagrange multiplier as defined in that section. The same argument used there shows that, given Condition 3, which are written in terms of κ given by (7), the Lagrangian is maximized at the proposed allocation $\boldsymbol{\pi}^*$. □