Trading on a Simplex: An Eaton–Kortum Diagram*

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Abstract

We present a deterministic re-interpretation of the Eaton and Kortum (2002) Ricardian model of international trade. The representation can be interpreted as a Dornbusch–Fischer–Samuelson generalization to an N-country world under specific technological restrictions that nest Eaton–Kortum and is amenable to diagrammatic analysis.

1 Introduction

The seminal paper of Eaton and Kortum (2002) (henceforth EK) builds on an earlier – also seminal – paper by Dornbusch, Fischer, and Samuelson (1977) (henceforth DFS). The DFS model is a Ricardian model with a continuum of goods, but only two countries. With only two countries, DFS are able to index goods by comparative advantage, rendering the equilibrium amenable to a transparent diagrammatic analysis. However, with more than two countries comparative advantage is a multi-dimensional object, and the diagrammatic analysis of DFS is lost (see Eaton and Kortum, 2002, footnote 13). EK elegantly circumvent this obstacle by taking a *probabilistic* approach to characterize an N-country Ricardian model with a continuum of goods. Specifically, by assuming that productivity is drawn independently across goods and countries from a Fréchet distribution with common shape parameter, and exploiting the properties of the Fréchet and a continuum of goods, EK obtain a gravity representation of bilateral trade and other compact general equilibrium conditions. This allows for the analysis of a rich multi-country model that can

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be taken to the data. However, the useful graphical intuition one obtains from DFS appears out of reach. This paper introduces a diagrammatic approach to analyzing EK that is reminiscent of the original DFS framework. Rather than using a probabilistic structure for technology, we map a deterministic commodity space into a simplex. We show how the simplex allows us to characterize comparative advantage and the assumptions under which trade shares are determined by the same gravity formulas as in EK. We can then perform comparative static exercises to graphically explore how the pattern of trade changes with trade costs or productivity, as in the original DFS.

We proceed by introducing a simple Ricardian model with a continuum of goods and show how to make it amenable to diagrammatic analysis. We then show that the core equilibrium conditions of this model coincide with those of the original EK model. Given that Eaton and Kortum (2002) already proposed a 2-country version (see their footnote 15), our benchmark below is an $N = 3$ country world. We show how it is straightforward to extend it to any N.

2 Environment

We consider a static environment with $N = 3$ countries, indexed by $i \in \{1, 2, 3\}$. It is straightforward to generalize to $N > 3$ countries, which we do after discussing the benchmark case. Labor is the only factor of production, and country i is endowed with L_i units of labor.

2.1 Technology and Goods

As in DFS, there is a unit-measure continuum of goods. With two countries, DFS are able to sort goods by comparative advantage in a line (that is, relative productivity of the good in the two countries). With three countries, comparative advantage becomes multi-dimensional. We let goods be indexed by their vector of labor productivities in each country, $(z_1, z_2, z_3) \in \mathbb{R}^3_{++}$.

The key step in our approach is to map the space of productivities $\mathbb{R}^3_{++}$ into a space more amenable to analysis. For a given $\theta > 0$ and $T_i > 0$, let $y_i = T_i z_i^{-\theta}$ and let us define the following transformation of coordinates that maps $\mathbb{R}^3_{++}$ into a *comparative advantage vector* $\omega = (a_1, a_2, a_3)$ in the simplex $\Omega \equiv \{a \in \mathbb{R}^3_+ : a_1 + a_2 + a_3 = 1\}$ and a *world productivity* $s \in \mathbb{R}_{++}$:

$$a_i = \frac{y_i}{\sum_k y_k}, \qquad s = \left(\sum_k y_k \right)^{-1/\theta}.$$

In this transformation, the value of a_i tracks the comparative advantage of producing the good in country i: a low value of a_i means country i is relatively productive in it. The world productivity s tracks how productive all countries are in producing the good: it scales the good's productivity in every country. To see this, we can invert the transformation. Let $A_i \equiv T_i^{1/\theta}$, then

country i's productivity in a good with comparative advantage ω and world productivity s is

$$z_i = A_i \, s \, a_i^{-1/\theta},$$

which shows that s scales the productivity of all countries in producing this good (and A_i scales the productivity of all goods from country i).[1]

Note that we can always do such a change of coordinates for any θ and any $\{T_i\}$. This change maps the unit mass of goods into the set $\mathcal{N} \equiv \Omega \times \mathbb{R}_{++}$, and we let $\mu(G)$ denote the mass of goods that are mapped into set $G \subseteq \mathcal{N}$. We index a good by $\nu = (\omega, s) \in \mathcal{N}$ and work directly in this space.

The first restriction in the commodity space is that there exists a transformation that induces a uniform measure on the simplex:[2]

Assumption 1 (Uniformity on the Simplex). *There exists a transformation with $\theta > 0$ and $T_1, T_2, T_3 > 0$ such that the fraction of goods whose comparative-advantage vectors lie in any region of Ω equals that region's share of the simplex's area.*

To visualize what this assumption imposes, we introduce the diagram shown in Figure 1, which will become our main expository tool. The edges of the depicted equilateral triangle with height 1 correspond to the three countries $i \in \{1, 2, 3\}$. Each point inside the triangle represents goods with comparative advantage vector $\omega = (a_1, a_2, a_3)$, where a_i is the perpendicular distance from the point to the edge i. Points closer to edge i represent goods for which country i is relatively productive (recall that a_i represents inverse relative productivity). For example, as ω gets closer to the edge 1, the productivity index a_1 approaches 0, implying that country 1's relative productivity goes to ∞. This means that goods with an ω close to the edge 1 are those goods where country 1 has a comparative advantage.[3]

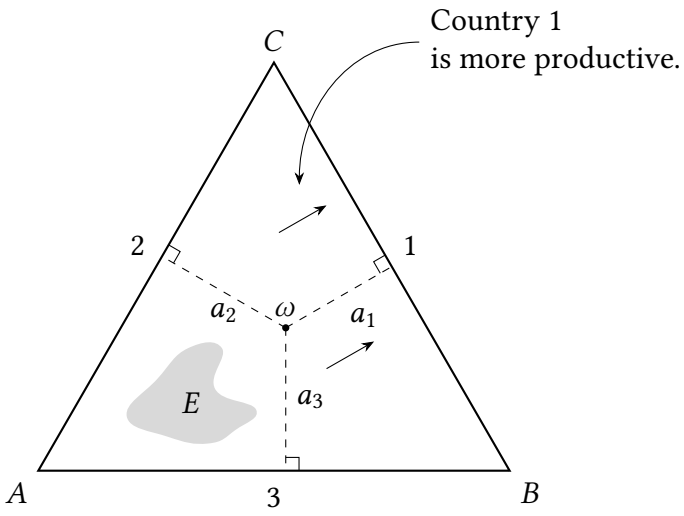
[1] Alvarez and Lucas (2007) describe a good by a vector of N values of y_i/T_i. They assume that these are independent exponentials with parameters T_i, and note that this model is EK's. Our analysis instead decomposes this vector into a comparative advantage vector and a world productivity level, and imposes assumptions that recover EK's expenditure shares. Section 6 describes the assumptions under which EK arises in our case.

[2] More technically, for a region $E \subseteq \Omega$, let $|E|$ denote its area. Assumption 1 states that

$$\mu(E \times \mathbb{R}_{++}) = \frac{|E|}{|\Omega|} \qquad \text{for every region } E \subseteq \Omega.$$

[3] Simplexes have appeared before in trade theory to discuss factor price equalization (McKenzie, 1955; Leamer, 1987). Wilkens (2009) and Shiozawa (2015) both use a simplex to study comparative advantage in a Ricardian model without trade costs, where vertices are countries, there are a finite number of goods, and a good's location in the simplex represents its relative productivities. Our simplex uses reciprocal coordinates: a point represents the relative inverse productivities (to the power θ) of a good. The continuum of goods and the measure on the simplex have no

Figure 1: The Comparative Advantage Simplex



Note: Under the uniformity assumption, Assumption 1, the fraction of goods whose comparative-advantage vectors lie in a region such as E equals that region's share of the simplex's area.

Assumption 1 requires that there exists a transformation $(\theta, \{T_i\})$ such that the resulting comparative advantage vectors are uniformly distributed on the simplex, as illustrated by the set E in Figure 1: the fraction of goods with comparative advantage vectors in any such set is exactly the fraction of the simplex area that is occupied by that set. This is a strong assumption: it is easy to construct cases where it won't hold. In addition, when Assumption 1 holds, it does so for a unique θ. That is, a technology can be compatible with at most one θ. (The values of T_i are also unique up to a common factor.)

We show below that Assumption 1 implies that the fraction of goods sourced from different locations takes the EK gravity form. The next assumption strengthens Assumption 1 and is the one that guarantees that expenditure shares take the same gravity form as the fraction of goods sourced (given the preferences we specify below):[4]

counterparts in these previous works.

[4]With $|E|$ as in footnote 2, Assumption 2 states that

$$\mu(E \times B) = \frac{|E|}{|\Omega|}\,\mu(\Omega \times B) \qquad \text{for every region } E \subseteq \Omega \text{ and set of values of world productivity } B \subseteq \mathbb{R}_{++}.$$

Setting $B = \mathbb{R}_{++}$ gives Assumption 1.

Assumption 2 (Uniformity on the Simplex Given World Productivity)**.** *There exists a transformation with $\theta > 0$ and $T_1, T_2, T_3 > 0$ such that for any s, the fraction of goods among those with world productivity s whose comparative-advantage vectors lie in any region of Ω equals that region's share of the simplex's area.*

From the perspective of Figure 1, Assumption 2 requires that the uniformity applies conditional on s. That is, among goods with any given world productivity s, the comparative advantage vectors remain uniformly distributed on the simplex.

The final technological assumption is the standard iceberg transport cost: delivering one unit of a good from i to j requires shipping $\tau_{ij} \geq 1$ units.

2.2 The Firm's Problem

Given the technology description above, firms are competitive and hire labor at a wage w_i in country $i = 1, 2, 3$. The cost of producing good $v = (\omega, s) \in \mathcal{N}$ in country i for delivery in country j is:

$$\frac{\tau_{ij} a_i^{1/\theta} w_i}{A_i s}.$$

Competition implies that the price of good v in country j is the marginal cost of the minimum-cost supplier:

$$p_j(v) = \min_i \left\{ \frac{\tau_{ij} a_i^{1/\theta} w_i}{A_i s} \right\} = \frac{1}{s} \min_i \left\{ \frac{\tau_{ij} a_i^{1/\theta} w_i}{A_i} \right\}, \tag{1}$$

where we have taken the factor $1/s$ outside the minimization, as it is the same for all sources. Hence, which country is the cheapest supplier of a good $v = (\omega, s)$ depends only on ω, the good's comparative advantage vector.

2.3 The Household's Problem

The representative household in country i has preferences characterized by the utility function $U_i\left(\{c_i(v)\}_{v \in \mathcal{N}}\right)$. To determine the pattern of production we do not need to say more about preferences at this stage. Later on, we impose a type of symmetry restriction on preferences across goods.

The household's problem in country i is:

$$\max_{\{c_i(v)\}} U_i\left(\{c_i(v)\}\right) \text{ subject to } X_i \equiv \int_{\mathcal{N}} p_i(v) c_i(v)\, d\mu(v) \leq w_i L_i,$$

where we define X_i to be the total expenditure of country i.

Let $\Omega_{ji} \subseteq \Omega$ denote the set of comparative advantage vectors for goods that country i purchases from country j (as we discussed above, this choice is only a function of ω). That is, $\omega \in \Omega_{ji}$ if $j = \arg\min_k \left\{\tau_{ki} a_k^{1/\theta} w_k / A_k\right\}$, where we ignore the measure-zero set of ties. Let X_{ji} denote the expenditure of country i on goods from country j:

$$X_{ji} = \int_{\Omega_{ji} \times \mathbb{R}_{++}} p_i(v) c_i(v)\, d\mu(v).$$

Summing across sources, $\sum_j X_{ji} = X_i$.

2.4 Equilibrium

The definition of equilibrium is standard:

Definition 1. *An* equilibrium *is a set of prices and consumption in each country* $\{p_i(v), c_i(v)\}$ *for* $v \in \mathcal{N}$*, and a vector of wages* $\{w_i\}$*, such that households optimize given prices and wages and spend their labor income,* $X_i = w_i L_i$*; prices satisfy* (1)*; and global expenditure on goods sourced in* i *equals labor income in* i*:* $\sum_j X_{ij} = w_i L_i$.

3 A Graphical Representation in the Simplex

We now show how the pattern of production can be represented graphically using the simplex introduced in Figure 1. In what follows we highlight the roles of Assumptions 1 and 2 in generating the results.

3.1 Comparative Advantage and Sourcing

We focus first on the sourcing decision, given wages. In the $N = 2$ case, DFS define a "marginal good" where the cost of supplying a good to a given country is the same when sourced from the two countries. In our $N = 3$ case, we define instead a locus of comparative advantage vectors in the simplex.

Specifically, let $\{w_i\}$ be a vector of equilibrium wages. Consider a consumer in country 1. Let $M^1(i, j)$ be the set of comparative advantage vectors for which the cost is the same whether the

good is produced in i or j:

$$M^1(i,j) = \left\{ \omega = (a_1, a_2, a_3) \in \Omega \text{ such that } \frac{\tau_{i1} w_i a_i^{1/\theta}}{A_i} = \frac{\tau_{j1} w_j a_j^{1/\theta}}{A_j} \right\}.$$

$M^1(i,j)$ is then the set of points ω for which the ratio a_i/a_j is a constant:

$$\frac{a_i}{a_j} = \left(\frac{\tau_{j1} w_j / A_j}{\tau_{i1} w_i / A_i} \right)^{\theta}.$$

Given the proportionality of a_i/a_j, the set $M^1(i,j)$ is a line in the simplex. For example, $M^1(2,3)$ is a ray from the vertex A where the edges 2 and 3 meet, as depicted in Figure 2. Similar rays from the other vertices correspond to $M^1(1,2)$ and $M^1(1,3)$. In Figure 2 these loci intersect at the point labeled O. For a good with comparative advantage O, country 1 faces the same cost regardless of source country. On the interior of the triangle AOB whose base is edge 3 and has edges AO and BO, country 1 will source from country 3. Similarly for the other two partitions. Keep in mind that point O and the associated threshold rays are drawn from the perspective of country 1 consumers; with trade costs, these objects will vary across alternative destination countries.

In anticipation, we flag the importance of the point O. As will become clear, key equilibrium objects such as expenditure shares and the price distribution will be indexed by this point. Similarly, many comparative statics will follow from understanding how O moves in response.

The expression for the coordinates of point O has a familiar form. At point $O = (a_1^O, a_2^O, a_3^O)$, we have

$$a_2^O = \left(\frac{\tau_{11} w_1 / A_1}{\tau_{21} w_2 / A_2} \right)^{\theta} a_1^O, \qquad a_3^O = \left(\frac{\tau_{11} w_1 / A_1}{\tau_{31} w_3 / A_3} \right)^{\theta} a_1^O.$$

The fact that $a_1 + a_2 + a_3 = 1$ implies

$$a_1^O = \frac{(\tau_{11} w_1 / A_1)^{-\theta}}{\sum_{j=1}^3 (\tau_{j1} w_j / A_j)^{-\theta}}.$$

Similar expressions hold for a_2^O and a_3^O. In particular, for $i = 1, 2, 3$, we have:

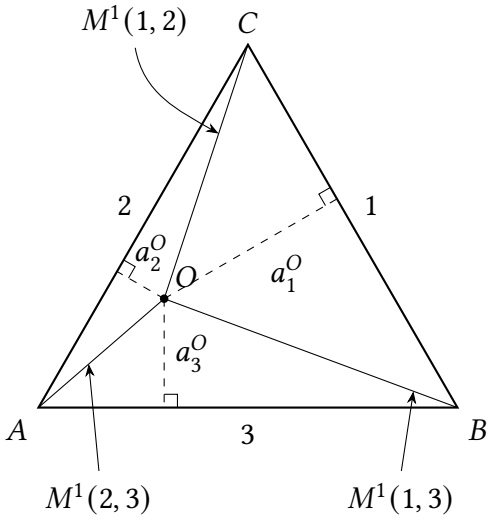
$$a_i^O = \frac{(\tau_{i1} w_i / A_i)^{-\theta}}{\sum_{j=1}^3 (\tau_{j1} w_j / A_j)^{-\theta}}. \tag{2}$$

The expression (2) resembles the standard CES expenditure share formula, but importantly we have not imposed any such restriction on preferences in the derivation or linked it to any notion

of expenditure.

The role of Assumption 1. Consider the goods sourced from country 3; that is, triangle AOB. The area of this triangle is $a_3^O \times \overline{AB}/2$, where $\overline{AB}$ is the length of the base of AOB. As all sides of ABC have the same length, the total area is $\overline{AB} \times \sum_i a_i^O/2 = \overline{AB}/2$. By Assumption 1, a_3^O represents the fraction of goods that country 1 sources from country 3. The areas of the triangles AOC and BOC then determine the fractions of goods sourced from countries 2 and 1, respectively (that is, a_2^O and a_1^O). Under Assumption 1, expression (2) equals the fraction of goods that country 1 sources from country i.

Figure 2: The Threshold Goods



A similar analysis can be done for countries 2 and 3. The respective lines $M^2(i,j)$ and $M^3(i,j)$ will determine the intersection points O for each destination country.

3.2 Prices by Source

Recall that goods are indexed by two components ω and s, that is, $v = (\omega, s)$. To characterize prices, we restrict attention to the case in which goods have the same world productivity. We normalize this constant world productivity to $s = 1$ (which can be done by rescaling T_i). Note that in this case, Assumption 2 holds automatically under Assumption 1.

Given that $s = 1$, every point in the simplex represents just one good, good $(\omega, 1)$. Point O is then the good for which all potential source countries have the same cost of supplying country 1, and it is the good with the highest price in country 1, which we denote $\overline{p}_1$. To see this, note

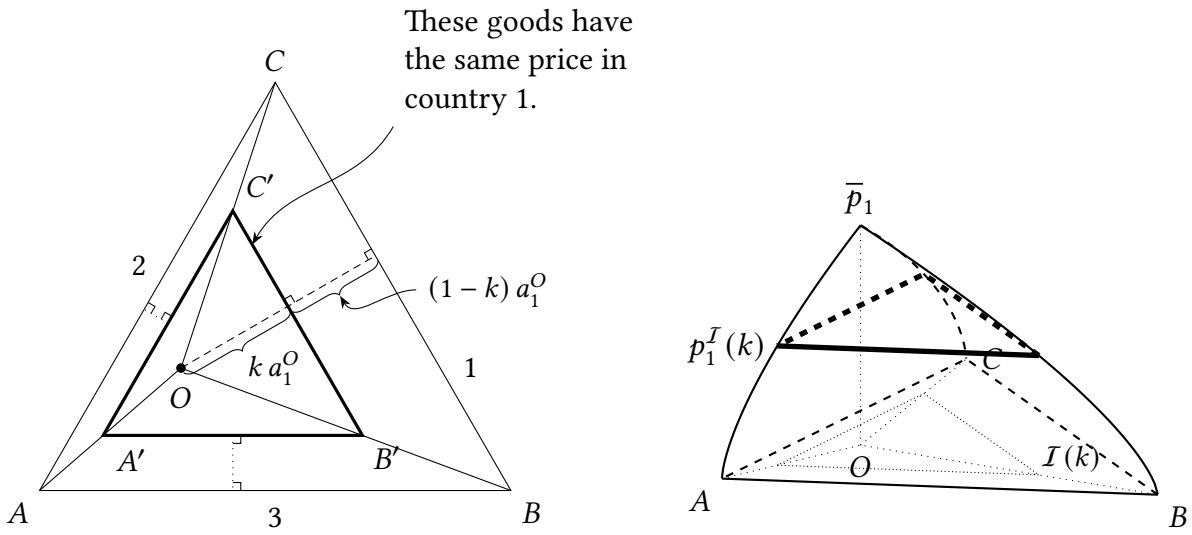
that every good ω has at least one source country i with $a_i \leq a_i^O$; which implies that that country can supply the good at a cost weakly lower than at O. Hence

$$p_1(\omega, 1) = \min_k \frac{\tau_{k1} a_k^{1/\theta} w_k}{A_k} \leq \frac{\tau_{i1} a_i^{1/\theta} w_i}{A_i} \leq \frac{\tau_{i1} (a_i^O)^{1/\theta} w_i}{A_i} = \overline{p}_1 \quad \text{for all } \omega \in \Omega.$$

Consider the triangle $A'B'C'$ depicted in Panel (a) of Figure 3. The edge $A'B'$ is parallel to AB; hence, goods on this edge are produced by country 3 and all have the same cost for consumption in country 1 (recall that the distance to the edge is inversely related to productivity). Similarly for the other edges of the triangle. At the vertices, costs are the same for the respective pair of countries. Consumers in country 1 therefore face the same price for all goods along the edge of this triangle. We call this triangle an *iso-price* triangle. On the interior of this triangle, goods are more expensive (there is less comparative advantage), and outside the triangle goods are cheaper.

In Panel (b) we show a three-dimensional representation of the "price pyramid" with the simplex of goods as the base and the height at each point the price of the good. The maximum is $\overline{p}_1$, the price at point O, and prices are zero along the edges of the simplex as productivity goes to infinity. The vertical edges of the price pyramid are depicted for $\theta = 1.5$ and hence are slightly concave relative to the base.

Figure 3: Iso-Price Loci when $s = 1$



(a) An iso-price triangle in the simplex

(b) The price pyramid

We can index the iso-price triangles by a scaling factor $k \in [0, 1]$. Suppose the triangle has distance to the edges of the full simplex given by $(1 - k) \times (a_1^O, a_2^O, a_3^O)$; that is, in Figure 3, the perpendicular distance between $A'B'$ and AB is $(1 - k)a_3^O$, and similarly for the other edges. The

highest price, $\overline{p}_1$, can be expressed as[5]

$$\overline{p}_1 = \left(\sum_i \left(\tau_{i1} w_i / A_i \right)^{-\theta} \right)^{\frac{-1}{\theta}}. \tag{3}$$

Consider now a good along a boundary of $A'B'C'$, say on $B'C'$. This good has $a_1 = (1-k)a_1^O$. It follows that the price of this good with $s = 1$ (which will be produced in country 1) is:

$$p_1(\omega, 1) = \frac{\tau_{11} \left((1-k) a_1^O \right)^{1/\theta} w_1}{A_1} = (1-k)^{1/\theta} \overline{p}_1.$$

This relationship applies to any point on the boundary of $A'B'C'$, as all goods on the boundary have the same price.

Let $\mathcal{I}(k)$ denote the boundary of this iso-price triangle:

$$\mathcal{I}(k) \equiv \left\{ \omega \in \Omega : p_1(\omega, 1) = (1-k)^{1/\theta} \overline{p}_1 \equiv p_1^{\mathcal{I}}(k) \right\}. \tag{4}$$

The fraction of goods with price higher than $p_1^{\mathcal{I}}(k)$ is the area of $A'B'C'$ relative to that of the simplex, and the relative areas of the three sub-triangles with vertex O represent the fraction of those goods sourced from each country. This iso-price triangle is depicted by the thick contour $p_1^{\mathcal{I}}(k)$ in Panel (b) of Figure 3. Its projection onto the simplex denoted by the dotted lines in panel (b) represents the set $\mathcal{I}(k)$.

As we vary k, we shrink or expand the iso-price triangle around point O, tracing out the distribution of prices for consumer 1. The areas sourced in each country shrink or expand with k, as well. Given the proportionality, the fraction sourced from each country is stable for each price. More generally, for any iso-price triangle, the fraction of goods with higher price from each country is the same as in the full price distribution. Thus, we obtain the well known result from Eaton and Kortum (2002): the distribution of prices is independent of the source country.

The role of Assumption 2. We have chosen to illustrate the result under the restriction $s = 1$, but let us briefly point out that it holds more generally under Assumption 2. To see this, note that a good with comparative advantage vector ω and world productivity s has price $p_1(\omega, s) = p_1(\omega, 1)/s$. Then, for any world productivity s, by Assumption 2, among goods with world productivity s and price above any p, a fraction a_i^O is sourced from country i. Integrating across different values of world productivity maintains this fraction and preserves the result that the

[5] To see this, note that $\overline{p}_1 = (a_i^O)^{1/\theta} \tau_{i1} w_i / A_i$, for any i as the costs are equal across sources at point O. Inverting, we have $a_i^O = \overline{p}_1^{\theta} (\tau_{i1} w_i / A_i)^{-\theta}$. Summing across i and using $\sum_i a_i^O = 1$, we can solve for $\overline{p}_1$ given by equation (3).

distribution of prices is independent of the source country. This also explains why Assumption 1 is not enough: even if the total projection onto Ω is uniform, regions of the simplex supplied by different countries may contain goods with different world productivity, and hence may feature different price distributions.

3.3 Demand and Expenditure Shares

So far we have not used any knowledge of consumer demand. In what follows, we will assume *that households spend the same amount on goods that have the same price and world productivity.* More formally:

Assumption 3 (Symmetry of Preferences)**.** *For each $i \in \{1, 2, 3\}$, at any solution to country i's household problem, goods with the same price and world productivity s receive the same consumption.*

Together with Assumption 2, this assumption is sufficient to recover the expenditure share formulas of Eaton and Kortum (2002). Note that no restriction is placed on the elasticity of substitution across goods, except that we need the household problem to have a solution.[6] In our benchmark with all goods sharing the same world productivity $s = 1$, this assumption is the same as requiring that households spend the same amount on goods with equal prices. When world productivity varies across goods, this assumption allows households to treat goods with different world productivity differently, as long as they do not distinguish goods by their comparative advantage vector ω.

Recall that for the pattern of production, we only need to know the comparative advantage vectors. We can then consider expenditure patterns one value of world productivity at a time. In Section 3.2 we showed that for a given world productivity s and a price p, a fraction a_i^O of the goods with a price above p is sourced from country i. This means that for any price interval $[p, p + \Delta)$, the share of goods sourced from country i is also a_i^O. By Assumption 3, all goods with the same price and world productivity receive the same expenditure. Hence, since this holds for every sub-interval of prices, a fraction a_i^O of country 1's spending on goods with prices in $[p, p + \Delta)$ and world productivity s goes to country i. As this is true for every price interval and every value of world productivity, adding up we get

$$X_{i1} = a_i^O X_1.$$

[6] For symmetric CES preferences with elasticity of substitution σ, this requires $\sigma < 1 + \theta$, as in Eaton and Kortum (2002).

Using the expression for a_i^O from (2), we have

$$\frac{X_{i1}}{X_1} \equiv \pi_{i1} = a_i^O = \frac{(\tau_{i1} w_i / A_i)^{-\theta}}{\sum_{j=1}^{3} (\tau_{j1} w_j / A_j)^{-\theta}}, \qquad \text{for all } i = 1, 2, 3.$$

Indeed now we have that the expenditure share on country i's goods is as if consumers had a CES demand, with $\tau_{i1} w_i / A_i$ representing the cost of sourcing from country i for consumption in country 1, and $1 + \theta$, the elasticity of substitution.[7] More generally, for every destination $d \in \{1, 2, 3\}$ we have:

$$\frac{X_{id}}{X_d} \equiv \pi_{id} = \frac{(\tau_{id} w_i / A_i)^{-\theta}}{\sum_{j=1}^{3} (\tau_{jd} w_j / A_j)^{-\theta}}, \qquad \text{for all } i = 1, 2, 3. \tag{5}$$

In terms of our graphical representation, going back to Figure 2, the fraction of the area of the simplex covered by the triangle BOC is the expenditure share of country 1 on its own goods, and similarly, the areas of AOC and AOB represent the expenditure shares on countries 2 and 3, respectively.

The role of Assumption 3. We have already seen that Assumption 2 makes the distribution of prices independent of the source country for any world productivity. The fact that the share of expenditure on country i is the same as the share of the *number* of goods from country i relies on demand: Assumption 3 implies that for any world productivity, expenditure on goods with the same price is the same, and thus the fraction of expenditure on goods from country i equals the fraction of goods sourced from that country.

3.4 Closing the Model

The analysis so far has been predicated on a vector of wages (w_1, w_2, w_3). To close the model, we need to ensure that the labor market clears and that trade is balanced:

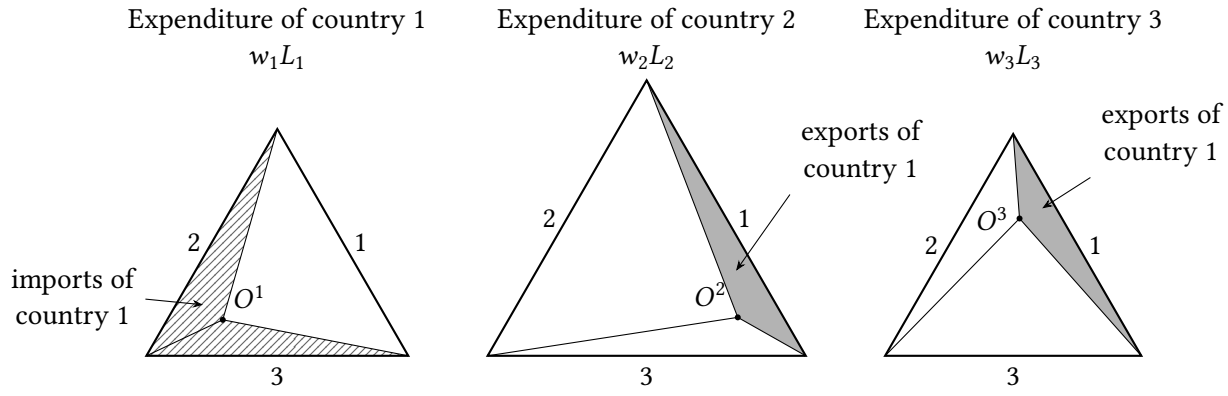
$$X_{i1} + X_{i2} + X_{i3} = w_i L_i = X_i, \qquad \text{for all } i = 1, 2, 3. \tag{6}$$

Equations (5) and (6) are sufficient to determine the equilibrium wages and expenditure shares (one of the equations is redundant given Walras' law, but recall that we also need to set a numeraire). On wages and prices we have one degree of freedom, and our numeraire will be country 1's wage: $w_1 = 1$. Note that the equilibrium conditions do not depend on world productivity s

[7] This also shows the equivalence of the Eaton and Kortum (2002) model with an Armington model with an elasticity of substitution given by $1 + \theta$.

beyond Assumption 2, nor on preferences beyond Assumption 3.

Figure 4: Balanced Trade for Country 1



Note: The total area of each simplex is scaled by w_iL_i, the total expenditure of country i. The hatched areas in the first triangle are the imports of country 1 from countries 2 and 3. The shaded areas in the other two triangles are the exports of country 1 to countries 2 and 3, respectively. Balanced trade for country 1 requires that the hatched area in the first triangle equal the sum of the shaded areas in the other two (imports equal exports).

Figure 4 displays the balanced-trade condition for country 1. Each triangle is a country's simplex, with its area scaled by the equilibrium w_iL_i, the total expenditure of that country. Balanced trade requires that country 1's expenditure on imports equal the value of its exports to countries 2 and 3. Hence the hatched regions must have the same total area as the shaded ones:

$$\underbrace{X_{21} + X_{31}}_{\text{hatched}} = \underbrace{X_{12} + X_{13}}_{\text{shaded}}.$$

A similar condition also holds for country 2 and country 3 expenditures and income.

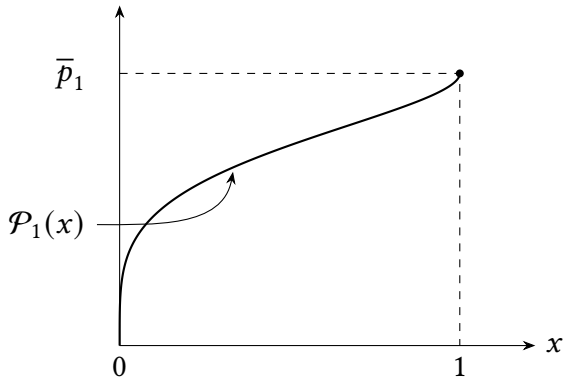
4 The Price Distribution and Welfare

We have already seen that demand does not matter for the equilibrium determination of the wages or expenditure shares, beyond the restriction imposed in Assumption 3. In this section, we use the simplex to characterize the price distribution faced by consumers in a country for the case where $s = 1$. This shows why the pattern of substitution in demand does not matter for welfare either (beyond the symmetry restriction on preferences).

Taking the perspective of country 1, recall that $\mathcal{I}(k)$ defined in (4) defines the locus of goods with price $p_1^{\mathcal{I}}(k) = (1-k)^{1/\theta}\overline{p}_1$. This set is the boundary of triangle $A'B'C'$ in Figure 3. Inside this triangle, $p_1(\omega, 1) > p_1^{\mathcal{I}}(k)$. As the iso-price triangle is similar to the simplex, but scaled down by

a factor k, it has area k^2 times the area of the simplex.[8] That is, the share of goods with prices above $p_1^{\mathcal{I}}(k)$ is k^2, and hence the share of goods with prices less than $p_1^{\mathcal{I}}(k)$ is $1-k^2$.

Figure 5: The Price Distribution



Following French (2015), it is convenient to invert the price distribution.[9] Specifically, $x = 1-k^2$ is the share of goods with price less than $p_1^{\mathcal{I}}(k)$. Inverting, $k = (1-x)^{1/2}$. Substituting into $p_1^{\mathcal{I}}(k) = (1-k)^{1/\theta}\overline{p}_1$, the price $\mathcal{P}_1(x)$ below which a share x of goods lies is

$$\mathcal{P}_1(x) = \overline{p}_1 \left[1-(1-x)^{1/2}\right]^{1/\theta}. \tag{7}$$

Figure 5 depicts such a price distribution. Note that the price distribution is scaled by $\overline{p}_1$, the price of good O. Just as for trade shares, the location of this point is all we need to know to evaluate the equilibrium from the perspective of a country 1 consumer, holding τ_{11}, w_1 and A_1 fixed.

In this spirit, we can evaluate comparative statics with respect to labor endowments L_i, productivity shifters A_i, or trade costs τ_{ij} by evaluating what happens to O. Given (7), such changes will shift the price distribution, but not alter the relative price between two goods at given ranks: in equilibrium, a change in labor endowments or trade costs always generates a proportional shift in the distribution of prices.

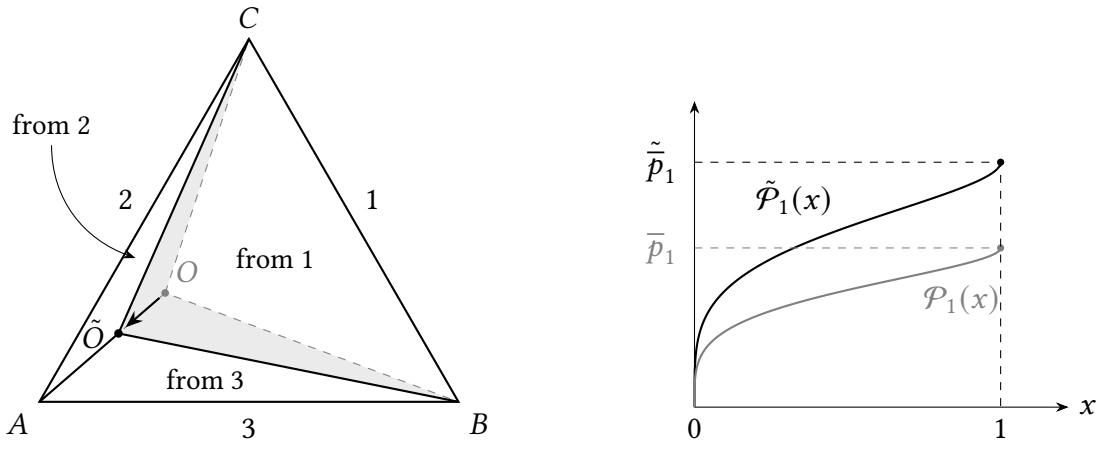
For example, consider an increase in trade costs from countries 2 and 3 into country 1; that is, τ_{i1}, $i = 2, 3$, increase. We depict the shift in Panel (a) of Figure 6. With fixed wages, an increase in trade costs shifts point O further away from country 1's edge. Country 1 spends more of its income on its own goods and imports less from 2 and 3. Moreover, $\overline{p}_1$ increases to $\tilde{\overline{p}}_1$.[10] Note

[8] Recall that similar triangles have sides of equal ratios. By definition, the perpendicular distance from the edge of $A'B'C'$ to O is k times a_i^O, where i is the respective edge. This implies that, for example, $A'B'$ has a length k times that of AB. As both the base and height of $A'B'C'$ are scaled by k the area of the iso-price triangle is k^2 times the simplex.

[9] We thank Dave Donaldson for this reference.

[10] The highest price for country 1 households is the cost of good O, which is, from footnote 5, $\overline{p}_1 = (a_1^O)^{1/\theta}\tau_{11}w_1/A_1$.

Figure 6: An Increase in Trade Costs



(a) Sourcing regions

(b) The shift in prices

that this is illustrative, as in equilibrium, the wages will change. Panel (b) of Figure 6 shows the resulting effect on the price distribution: the entire price distribution shifts up and $\tilde{\mathcal{P}}_1(x) = \kappa \mathcal{P}_1(x) > \mathcal{P}_1(x)$ at every quantile x with $\kappa = \tilde{\overline{p}}_1/\overline{p}_1$.

French (2015) obtained the welfare changes from analyzing the expenditure function and exploiting the proportionality of the price distribution. Following his approach, consider a change in $\{\tau_{ij}\}, \{A_i\}, \{w_i\}$ that moves the initial equilibrium of prices of country 1 from p_1 to a new $\tilde{p}_1$. Under Assumption 3, the expenditure function scales with $\overline{p}_1$.[11] That is,

$$\frac{e(\tilde{p}_1, u)}{e(p_1, u)} = \frac{\tilde{\overline{p}}_1}{\overline{p}_1}.$$

Given our numeraire $w_1 = 1$ and $\tau_{11} = 1$, this implies that $\overline{p}_1$ increases with the distance of O from country 1's edge.

[11] Let $e(p, u)$ denote the expenditure function of country 1 and consider a path of model-generated prices $p_1(t)$, driven by costs $c_k(t) = \tau_{k1}(t) w_k(t)/A_k(t)$ that move smoothly, as t goes from 0 to 1, from their initial to their new equilibrium values. Abusing notation, we let $p_1(t, v)$ represent the price of good v at t. At every t, among goods with world productivity s and price q, a fraction $a_k^O(t)$ is sourced from country k, and the price of each such good sourced from k grows at rate $\dot{c}_k(t)/c_k(t)$, except at sourcing boundaries, which have measure zero. The average price change among these goods is therefore

$$q \sum_k a_k^O(t) \frac{\dot{c}_k(t)}{c_k(t)} = q \frac{\dot{\overline{p}}_1(t)}{\overline{p}_1(t)},$$

where the equality follows by differentiating $\overline{p}_1(t) = \left(\sum_k c_k(t)^{-\theta}\right)^{-1/\theta}$. Assumption 3 implies that goods with the same world productivity and price receive the same Hicksian consumption. Thus, grouping goods by world productivity and price, Shephard's lemma gives

$$\frac{d}{dt} e(p_1(t), u) = \int c^h(t, v, u)\, \dot{p}_1(t, v)\, d\mu(v) = \frac{\dot{\overline{p}}_1(t)}{\overline{p}_1(t)} \int c^h(t, v, u)\, p_1(t, v)\, d\mu(v) = \frac{\dot{\overline{p}}_1(t)}{\overline{p}_1(t)} e(p_1(t), u).$$

Integrating from $t = 0$ to $t = 1$ yields $e(\tilde{p}_1, u)/e(p_1, u) = \tilde{\overline{p}}_1/\overline{p}_1$, where $p_1 = p_1(0)$ and $\tilde{p}_1 = p_1(1)$ are the initial and new equilibrium prices. This replicates French's (2015) result.

Let u_1 and $\tilde{u}_1$ denote utility in the initial and in the new equilibrium. Define the welfare change as the proportional change in income necessary to attain the new utility at the initial prices. That is,

$$\frac{e(p_1, \tilde{u}_1)}{e(p_1, u_1)} = \frac{e(\tilde{p}_1, \tilde{u}_1)\frac{\overline{p}_1}{\tilde{\overline{p}}_1}}{e(p_1, u_1)} = \frac{\tilde{w}_1 L_1/\tilde{\overline{p}}_1}{w_1 L_1/\overline{p}_1},$$

since $e(p_1, u_1) = w_1 L_1$ and $e(\tilde{p}_1, \tilde{u}_1) = \tilde{w}_1 L_1$. Note that the welfare change is independent of preferences beyond the symmetry assumption.

Given that $\tau_{11} = 1$, equation (5) implies that the share of country 1's expenditure on its own goods is:

$$\pi_{11} \equiv \frac{X_{11}}{X_1} = \frac{(w_1/A_1)^{-\theta}}{\sum_{j=1}^{3}(\tau_{j1} w_j/A_j)^{-\theta}} = \left(\frac{w_1}{A_1 \overline{p}_1}\right)^{-\theta},$$

so that $w_1/\overline{p}_1 = A_1 \pi_{11}^{-1/\theta}$. With L_1 and A_1 fixed, the proportional change in welfare between an initial equilibrium and a counterfactual one (tildes) is

$$\frac{\tilde{w}_1 L_1/\tilde{\overline{p}}_1}{w_1 L_1/\overline{p}_1} = \left(\frac{\tilde{\pi}_{11}}{\pi_{11}}\right)^{-1/\theta},$$

the well known Arkolakis, Costinot, and Rodríguez-Clare (2012) formula: the change in the own expenditure share and θ are sufficient for the welfare effects of any change in foreign labor endowments and trade costs.

One may still wonder why the substitution forces in demand do not matter at all for welfare comparisons in this Ricardian model. It is important to recall that the relative price between goods at any two given ranks is always unaffected, and hence substitution forces in demand do not enter into welfare comparisons.[12]

The analysis of this section restricted attention to $s = 1$. As we have already shown, when world productivity varies across goods, the price of good (ω, s) is $p_1(\omega, 1)/s$. So prices scale with $1/s$. Under Assumption 2, for goods with the same world productivity s, the price at quantile x within these goods is $\mathcal{P}_1(x)/s$. Prices remain proportional to $\overline{p}_1$, and the welfare results above are unchanged.

[12] Given that the price at every rank is proportional to $\overline{p}_1$, and hence relative prices do not adjust, the consumption basket is a Hicksian composite good with a price proportional to $\overline{p}_1$. We thank Kei-Mu Yi for bringing this to our attention.

5 Extension to N Countries

It is straightforward to extend the setup to include $N \neq 3$ countries. The case of $N = 2$ collapses to a version of the Dornbusch, Fischer, and Samuelson (1977) model. For the case of $N > 3$, besides the obvious adjustments to the formulas above, the only major change is to the price distribution, which becomes:

$$\mathcal{P}_1(x) = \overline{p}_1 \left[1 - (1-x)^{1/(N-1)}\right]^{1/\theta}.$$

6 The Mapping to Eaton-Kortum

In EK, the productivity of country i in each good is drawn independently across goods and countries from the Fréchet distribution

$$F_i(z) = e^{-T_i z^{-\theta}}, \qquad z > 0,$$

with $T_i = A_i^\theta$. These draws, once transformed, do not share a common world productivity level. However, the EK model is a special case of our commodity space. For $N = 3$, we obtain it by imposing Assumption 2 and further restricting the measure of goods across values of world productivity s to have the density

$$h(s) = \tfrac{\theta}{2} s^{-3\theta-1} e^{-s^{-\theta}}, \qquad s > 0.$$

It is straightforward to check that the resulting productivities in this case are those that are generated by the independent Fréchet productivity draws of EK.

Note that sourcing shares, expenditure shares, wages, and the welfare changes remain unchanged relative to the benchmark with $s = 1$, as variation in world productivity does not affect the equilibrium or the welfare results under Assumption 2. Only the distribution of prices changes. The mass of goods with price below q in country 1 becomes $1 - \exp\big(-(q/\overline{p}_1)^\theta\big)$, which is the distribution in Eaton and Kortum (2002).

The key to the EK results regarding expenditure shares, wages, and welfare changes is that the fraction of goods whose comparative advantage lies in any region of the simplex equals that region's share of its area, and that the world productivity of a good does not depend on its comparative advantage. That is Assumption 2.

7 Cobb–Douglas Preferences: The DFS Case

There is a case where Assumption 2 is not necessary for the gravity result, which returns us to the original DFS formulation. Suppose that the utility function of the households is symmetric and Cobb–Douglas, as in the original DFS. That is,

$$U_i\left(\{c_i(v)\}_{v\in\mathcal{N}}\right) = \int_{\mathcal{N}} \ln c_i(v)\, d\mu(v).$$

In this case, households spend the same amount on every good, $p_i(v)c_i(v) = X_i$ for every good v, independently of the good's price and of its world productivity. Assumption 1 implies that sourcing shares satisfy the gravity equation (2). Given that spending is the same on every good, it follows immediately that expenditure shares equal sourcing shares and also satisfy the gravity equation (5), without requiring Assumption 2.

Given that Assumption 2 has not been imposed, the price distribution need not be the same across different sources, as world productivity may vary with comparative advantage. The welfare results nevertheless still apply, given the unitary elasticity of substitution. To see this, note that utility is $\ln X_1 - \int_{\mathcal{N}} \ln p_1(v)\, d\mu(v)$, and since $p_1(\omega, s) = p_1(\omega, 1)/s$,

$$\int_{\mathcal{N}} \ln p_1(v)\, d\mu(v) = \int_{\mathcal{N}} \ln p_1(\omega, 1)\, d\mu(\omega, s) - \int_{\mathcal{N}} \ln s\, d\mu(\omega, s),$$

where the last term does not depend on costs. By Assumption 1, comparative-advantage vectors are uniformly distributed on Ω, so the distribution of $p_1(\omega, 1)$ across goods is the one characterized in (7), and it follows that $\int_{\mathcal{N}} \ln p_1(\omega, 1)\, d\mu(\omega, s)$ equals $\ln \overline{p}_1$ plus a constant. The change in welfare is therefore the change in $w_1 L_1/\overline{p}_1$, as before.

8 Conclusion

Eaton and Kortum (2002) write in their seminal paper that "Dornbusch, Fischer, and Samuelson work out the two-country case, but their approach does not generalize to more countries." In footnote 15, they describe how their new model can be mapped into a DFS model when restricted to only two countries. But they write in footnote 13 that "With more than two countries there is no such natural ordering of commodities." In this note, we have shown how this can be done by positing a simplex representation of technology.

In ongoing work we extend the model to incorporate the correlated productivity model of Lind and Ramondo (2023). This requires relaxing Assumptions 1 and 2.

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