Online appendix to "A Note on Interval Delegation"

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December 7, 2017

This appendix collects the proofs of Lemmas 1 through 4.

A Proof of Lemmas 1 - 4

In Proposition 1 of Amador and Bagwell (2013), conditions (c1), (c2), (c2′), (c3), and (c3′) all must hold for some specified $\gamma_L < \gamma_H$ in Γ in order for the (non-degenerate) interval allocation with bounds γ_L, γ_H to be optimal. Condition (c1) will be implied by our condition (Gc1). Let us restate the other conditions:

(c2) If $\gamma_H < \overline{\gamma}$,

$$(\gamma - \gamma_H)\kappa \geq \int_{\gamma}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma_H)) \frac{f(\tilde{\gamma})}{1 - F(\gamma)} d\tilde{\gamma}, \ \forall \gamma \in [\gamma_H, \overline{\gamma}]$$

with equality at γ_H.

(c2′) If $\gamma_H = \overline{\gamma}$, $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) \geq 0$.

(c3) If $\gamma_L > \underline{\gamma}$,

$$(\gamma - \gamma_L)\kappa \leq \int_{\underline{\gamma}}^{\gamma} w_\pi(\tilde{\gamma}, \pi_f(\gamma_L)) \frac{f(\tilde{\gamma})}{F(\gamma)} d\tilde{\gamma}, \ \forall \gamma \in [\underline{\gamma}, \gamma_L]$$

with equality at γ_L.

(c3′) If $\gamma_L = \underline{\gamma}$, $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$.

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Now we proceed with new definitions. First, in the case that $\hat{\gamma} \notin \Gamma$, extend π_f from the domain Γ to the domain $\Gamma \cup \{\hat{\gamma}\}$ by letting $\pi_f(\hat{\gamma}) = \pi^*$. Recall the discussion after Equation (5), that $\gamma = -b'(\pi_f(\gamma))$ for all $\gamma \in \Gamma \cup \{\hat{\gamma}\}$ and that $\pi_f(\hat{\gamma}) = \pi^*$ by construction when $\hat{\gamma} \in \Gamma$. It is the case that π_f is strictly increasing over this extended domain, and so the sign of $\gamma - \hat{\gamma}$ is the same as the sign of $\pi_f(\gamma) - \pi_f(\hat{\gamma})$.

Next, define the functions $G : \Gamma \to \mathbb{R}$ and $H : \Gamma \to \mathbb{R}$ as follows.

$$G(\gamma) \equiv \int_{\gamma}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma)) f(\tilde{\gamma}) d\tilde{\gamma} \tag{12}$$

$$H(\gamma) \equiv \int_{\underline{\gamma}}^{\gamma} w_\pi(\tilde{\gamma}, \pi_f(\gamma)) f(\tilde{\gamma}) d\tilde{\gamma}, \tag{13}$$

The following lemma summarizes some properties of G and H, which will be useful for later reference.

Lemma 5. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds. Then the functions G and H are continuous, with*

(a) *(i) $G(\overline{\gamma}) = 0$, and (ii) $H(\underline{\gamma}) = 0$.*

(b) *(i) $G'(\overline{\gamma}) = -w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) f(\overline{\gamma})$, and (ii) $H'(\underline{\gamma}) = w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) f(\underline{\gamma})$.*[1]

(c) *$H(\gamma) + G(\gamma)$ has the same sign as $\pi^* - \pi_f(\gamma)$.*

(d) *(i) $G(\underline{\gamma})$ has the same sign as $\pi^* - \pi_f(\underline{\gamma})$, and (ii) $H(\overline{\gamma})$ has the same sign as $\pi^* - \pi_f(\overline{\gamma})$.*

(e) *(i) G is weakly convex, and (ii) H is weakly concave.*

(f) *For any $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma \cup \{\hat{\gamma}\}$,*

$$G(\gamma) = \int_{\gamma}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma_0)) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma - \gamma_0)\kappa(1 - F(\gamma)) \tag{14}$$

$$H(\gamma) = \int_{\underline{\gamma}}^{\gamma} w_\pi(\tilde{\gamma}, \pi_f(\gamma_0)) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma - \gamma_0)\kappa F(\gamma). \tag{15}$$

Proof of Lemma 5. Continuity as well as parts *(a)* and *(b)* are straightforward.

To show parts *(c)* and *(d)*, note that strict concavity of w in its second term implies that $\int_{\underline{\gamma}}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi) f(\tilde{\gamma}) d\tilde{\gamma}$ is strictly decreasing in π. The action π^* satisfies the first-order condition (4), that $\int_{\underline{\gamma}}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) d\tilde{\gamma} = 0$. Therefore, $\int_{\underline{\gamma}}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi) f(\tilde{\gamma}) d\tilde{\gamma}$ has the sign of $\pi^* - \pi$. Part

[1] Take $G'(\overline{\gamma})$ as the left-derivative and $H'(\underline{\gamma})$ as the right-derivative at these points.

(c) then follows from observing that $H(\gamma)+G(\gamma)=\int_{\underline{\gamma}}^{\overline{\gamma}} w_\pi(\tilde{\gamma},\pi_f(\gamma))f(\tilde{\gamma})d\tilde{\gamma}$. Part *(d)* (i) and (ii) are special cases of part *(c)*, respectively applying part *(a)* (ii) and (i).

To show part *(e)*, observe that for w of the form (6) it holds that

$$w_\pi(\tilde{\gamma},\pi_f(\gamma)) = A[b'(\pi_f(\gamma)) + C(\tilde{\gamma})] = A[C(\tilde{\gamma}) - \gamma], \tag{16}$$

where the last equality follows from the FOC $b'(\pi_f(\gamma)) + \gamma = 0$. Plugging $w_\pi(\tilde{\gamma},\pi_f(\gamma)) = A[C(\tilde{\gamma}) - \gamma]$ into $G(\gamma)$ as defined in (12) and taking the derivative yields

$$\begin{aligned} G'(\gamma) &= -A[C(\gamma)-\gamma]f(\gamma) - \int_\gamma^{\overline{\gamma}} Af(\tilde{\gamma})d\tilde{\gamma} \\ &= -w_\pi(\gamma,\pi_f(\gamma))f(\gamma) - A(1-F(\gamma)) \\ &= [AF(\gamma) - w_\pi(\gamma,\pi_f(\gamma))f(\gamma)] - A \end{aligned}$$

which is nondecreasing by (Gc1). Therefore G is weakly convex. Similarly, plugging $w_\pi(\tilde{\gamma},\pi_f(\gamma)) = A[C(\tilde{\gamma}) - \gamma]$ into $H(\gamma)$ as defined in (13) and taking the derivative yields

$$\begin{aligned} H'(\gamma) &= A[C(\gamma)-\gamma]f(\gamma) - \int_{\underline{\gamma}}^{\gamma} Af(\tilde{\gamma})d\tilde{\gamma} \\ &= w_\pi(\gamma,\pi_f(\gamma))f(\gamma) - AF(\gamma) \end{aligned}$$

which is nonincreasing by (Gc1). Therefore H is weakly concave.

To show part *(f)*, first note from (16) that $w_\pi(\tilde{\gamma},\pi_f(\gamma)) = A[C(\tilde{\gamma})-\gamma]$ and that $w_\pi(\tilde{\gamma},\pi_f(\gamma_0)) = A[C(\tilde{\gamma})-\gamma_0]$ for any γ and γ_0. Combining these two equations, $w_\pi(\tilde{\gamma},\pi_f(\gamma)) = w_\pi(\tilde{\gamma},\pi_f(\gamma_0)) - A[\gamma-\gamma_0]$. Substituting this identity into (12) and (13) and integrating out $A[\gamma-\gamma_0]$ yields (14) and (15), for $\kappa = A$. □

The functions G and H will essentially take the place of the "forward" and "backward" biases from Alonso and Matouschek (2008). Specifically, after flipping the sign of both functions, G generalizes the forward bias, and H generalizes the backward bias. In Alonso and Matouschek (2008), the convexity of the backward bias, and the corresponding concavity of the forward bias, are important for establishing optimality of interval delegation. After the sign changes, that translates to the convexity of G and the concavity of H in Lemma (5) part *(e)*.[2]

Putting together parts *(a)*, *(b)*, *(d)*, and *(e)* of Lemma 5 yields the following result:

Lemma 6. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds.*

[2] Letting $\pi_P(\gamma)$ indicate the principal's preferred action at state γ, and taking the agent's preferred action $\pi_f(\gamma)$ to be linear in γ, Alonso and Matouschek (2008) define the forward bias $S(\gamma)$ and backward bias $T(\gamma)$

(i) If $\pi^* < \pi_f(\underline{\gamma})$, then $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) < 0$.

(ii) If $\pi^* = \pi_f(\underline{\gamma})$, then $H(\gamma) < 0$ for all $\gamma > \underline{\gamma}$.

(iii) If $\pi^* = \pi_f(\overline{\gamma})$, then $G(\gamma) > 0$ for all $\gamma < \overline{\gamma}$.

(iv) If $\pi^* > \pi_f(\overline{\gamma})$, then $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$.

Proof of Lemma 6. *(i)* If $\pi^* < \pi_f(\underline{\gamma})$ then $G(\underline{\gamma}) < 0$ (Lemma 5 part *(d)*(i)); and $G(\overline{\gamma}) = 0$ (Lemma 5 part *(a)*(i)). Therefore by convexity of G (Lemma 5 part *(e)*(i)), it must be that $G'(\overline{\gamma}) > 0$. Hence, by Lemma 5 part *(b)*(i), $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) < 0$.

(ii) If $\pi^* = \pi_f(\underline{\gamma})$ then $G(\underline{\gamma}) = 0$ (Lemma 5 part *(d)*(i)) and $G(\overline{\gamma}) = 0$ (Lemma 5 part *(a)*(i)). Therefore, by convexity of G (Lemma 5 part *(e)*(i)), it holds that $G(\gamma) = 0$ for all γ. The result then follows from Lemma 5 part *(c)*.

(iii) If $\pi^* > \pi_f(\overline{\gamma})$ then $H(\overline{\gamma}) > 0$ (Lemma 5 part *(d)*(ii)); and $H(\underline{\gamma}) = 0$ (Lemma 5 part *(a)*(ii)). Therefore by concavity of H (Lemma 5 part *(e)*(ii)), it must be that $H'(\underline{\gamma}) > 0$. Hence, by Lemma 5 part *(b)*(ii), $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$.

(iv) If $\pi^* = \pi_f(\overline{\gamma})$ then $H(\overline{\gamma}) = 0$ (Lemma 5 part *(d)*(ii)) and $H(\underline{\gamma}) = 0$ (Lemma 5 part *(a)*(ii)). Therefore, by concavity of H (Lemma 5 part *(e)*(ii)), it holds that $H(\gamma) = 0$ for all γ. The result then follows from Lemma 5 part *(c)*.

□

We now proceed to prove Lemma 1 from the main text.

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as

$$S(\gamma) \equiv (1 - F(\gamma))\pi_f(\gamma) - \int_\gamma^{\overline{\gamma}} \pi_P(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma}$$
$$T(\gamma) \equiv F(\gamma)\pi_f(\gamma) - \int_{\underline{\gamma}}^{\gamma} \pi_P(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma}.$$

While Alonso and Matouschek (2008) focus on the convexity of T, linearity of π_f makes that equivalent to the concavity of S.

Under the functional form (6), we can plug (16) into (12) and (13) to get that

$$G(\gamma) = \int_\gamma^{\overline{\gamma}} A(C(\tilde{\gamma}) - \gamma) f(\tilde{\gamma}) d\tilde{\gamma} = -A\left((1 - F(\gamma))\gamma - \int_\gamma^{\overline{\gamma}} C(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma}\right)$$
$$H(\gamma) = \int_{\underline{\gamma}}^{\gamma} A(C(\tilde{\gamma}) - \gamma) f(\tilde{\gamma}) d\tilde{\gamma} = -A\left(F(\gamma)\gamma - \int_{\underline{\gamma}}^{\gamma} C(\tilde{\gamma}) f(\tilde{\gamma}) d\tilde{\gamma}\right).$$

As described in footnote 3 in the main text, the problem of Alonso and Matouschek (2008) can be transformed to one with utility of the form (6) in which $\pi_f(\gamma) = \gamma$, $A = 1$, and $C(\gamma) = \pi_P(\gamma)$. The above expressions then imply that $G(\gamma) = -S(\gamma)$ and $H(\gamma) = -T(\gamma)$.

Proof of Lemma 1. The optimal allocation follows as an immediate application of Proposition 1 of Amador and Bagwell (2013) with $\gamma_L = \underline{\gamma}$ and $\gamma_H = \overline{\gamma}$, noting that (c1) is implied by (Gc1); (c2) holds vacuously; (c2′) holds by assumption; (c3) holds vacuously; and (c3′) holds by assumption. The fact that $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma})]$ follows from Lemma 6 parts *(i)* and *(iv)*. $\square$

To prove Lemmas 2 through 4, let us now write "relaxed" versions of (c2), (c3), (d2), and (d3). The relaxed version of (c2) just confirms that the condition holds with equality at $\gamma = \gamma_H$, rather than additionally checking weak inequality at $\gamma > \gamma_H$; call this (Rc2). Likewise, call (Rc3) the relaxation of (c3) to just hold with equality at $\gamma = \gamma_L$. Call (Rd2) and (Rd3) the relaxations of (d2) and (d3) in which the relevant inequalities hold only at $\hat{\gamma}$, and only when $\hat{\gamma}$ is on the interior of Γ. These new conditions can be written in terms of G and H.

(Rc2) If $\gamma_H < \overline{\gamma}$ in Γ, $G(\gamma_H) = 0$.

(Rc3) If $\gamma_L > \underline{\gamma}$ in Γ, $H(\gamma_L) = 0$.

(Rd2) If $\hat{\gamma} \in (\underline{\gamma}, \overline{\gamma})$, $G(\hat{\gamma}) \leq 0$.

(Rd3) If $\hat{\gamma} \in (\underline{\gamma}, \overline{\gamma})$, $H(\hat{\gamma}) \geq 0$.

In fact, each of the relaxed conditions will be sufficient to imply the original conditions.

Lemma 7. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds.*

(i) *Fixing $\gamma_H < \overline{\gamma}$ in Γ, (Rc2) implies (c2).*

(ii) *Fixing $\gamma_L > \underline{\gamma}$ in Γ, (Rc3) implies (c3).*

(iii) *(Rd2) implies (d2).*

(iv) *(Rd3) implies (d3).*

Part *(i)* is a restatement of a result in Lemma 1 of Amador and Bagwell (2016). The proof follows exactly as in that Lemma, relying on the convexity of G and its expression as (14).[3] The other parts extend similar arguments from the case of a cap at state γ_H to the cases of a floor at state γ_L, and to floors or caps at action π^*.

[3] More precisely, the proof of Lemma 1 of Amador and Bagwell (2016) defines a function $G(\gamma)$ directly as the expression for $G(\gamma)$ in (14), with $\gamma_0 = \gamma_H$. They show that (Gc1) implies convexity of this function over Γ, as in Lemma 5 part *(e)*(i) of the current paper, and that convexity implies the result.

Proof of Lemma 7. *(i)* Fix $\gamma_H < \overline{\gamma}$ and suppose that (Rc2) holds. Applying Equation (14) with $\gamma_0 = \gamma_H$, condition (c2) is equivalent to

$$G(\gamma) \leq 0 \text{ for } \gamma \geq \gamma_H \text{ in } \Gamma, \text{ with } G(\gamma_H) = 0.$$

It holds that $G(\overline{\gamma}) = 0$, and that $G(\gamma_H) = 0$ under (Rc2). So (c2) follows from convexity of G (Lemma 5 part *(e)*(i)).

(ii) Fix $\gamma_L > \underline{\gamma}$ and suppose that (Rc3) holds. Applying Equation (15) with $\gamma_0 = \gamma_L$, condition (c3) is equivalent to

$$H(\gamma) \geq 0 \text{ for } \gamma \leq \gamma_L \text{ in } \Gamma, \text{ with } H(\gamma_L) = 0.$$

It holds that $H(\underline{\gamma}) = 0$, and that $H(\gamma_L) = 0$ under (Rc3). So (c3) follows from concavity of H (Lemma 5 part *(e)*(ii)).

(iii) If $\hat{\gamma} \geq \overline{\gamma}$, then (d2) holds vacuously. If $\hat{\gamma} < \overline{\gamma}$, applying Equation (14) with $\gamma_0 = \hat{\gamma}$ shows that condition (d2) is equivalent to

$$G(\gamma) \leq 0 \text{ for } \gamma \geq \hat{\gamma} \text{ in } \Gamma.$$

It holds that $G(\overline{\gamma}) = 0$. So if $\hat{\gamma} \in (\underline{\gamma}, \overline{\gamma})$, then by convexity of G, (d2) is implied by $G(\hat{\gamma}) \leq 0$, which is the condition (Rd2). On the other hand, if $\hat{\gamma} \leq \underline{\gamma}$ (i.e., $\pi^* \leq \pi_f(\underline{\gamma})$), then by convexity of G, (d2) is implied by $G(\underline{\gamma}) \leq 0$; and $G(\underline{\gamma}) \leq 0$ holds by Lemma 5 part *(d)*(i).

(iv) If $\hat{\gamma} \leq \underline{\gamma}$, then (d3) holds vacuously. If $\hat{\gamma} > \underline{\gamma}$, applying Equation (15) with $\gamma_0 = \hat{\gamma}$, condition (d3) is equivalent to

$$H(\gamma) \geq 0 \text{ for } \gamma \leq \hat{\gamma} \text{ in } \Gamma.$$

It holds that $H(\underline{\gamma}) = 0$. So if $\hat{\gamma} \in (\underline{\gamma}, \overline{\gamma})$, then by concavity of H, (d3) is implied by $H(\hat{\gamma}) \geq 0$, which is the condition (Rd3). On the other hand, if $\hat{\gamma} \geq \overline{\gamma}$ (i.e., $\pi^* \geq \pi_f(\overline{\gamma})$), then by concavity of H, (d3) is implied by $H(\overline{\gamma}) \geq 0$; and $H(\overline{\gamma}) \geq 0$ by Lemma 5 part *(d)*(ii).

□

The proofs of Lemmas 2 - 4 apply Lemma 7 in order to show existence of caps or floors satisfying the relevant conditions out of (c2), (c2′) (c3), (c3′), (d2), or (d3). Lemma 7 tells us that we need only check the relaxed conditions. That is, we need only confirm equalities or inequalities at single points rather than over an entire interval. This observation allows us to use arguments from continuity – i.e., the intermediate value theorem – to find the existence of such points.

Proof of Lemma 2. It holds that $G(\overline{\gamma}) = 0$, and by (12) there exists γ arbitrarily close to $\overline{\gamma}$ such that $G(\gamma) < 0$.[4] Lemma 6 part *(iii)* therefore rules out $\pi^* = \pi_f(\overline{\gamma})$. Moreover, Lemma 6 part *(iv)* rules out $\pi^* > \pi_f(\overline{\gamma})$. So there are two possible cases:

(i) $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$.

(ii) $\pi^* \leq \pi_f(\underline{\gamma})$, i.e., $\hat{\gamma} \leq \underline{\gamma}$.

In case *(i)*, first observe that $H(\gamma) \leq 0$ for all γ; this follows from $H(\underline{\gamma}) = 0$ (Lemma 5 part *(a)*(ii)), $H'(\underline{\gamma}) \leq 0$ (Lemma 5 part *(b)*(ii)), and H concave (Lemma 5 part *(e)*(ii)). In particular, $H(\hat{\gamma}) \leq 0$. So by Lemma 5 part *(c)*, it must hold that $G(\hat{\gamma}) \geq 0$. Therefore, continuity of G implies that there exists $\gamma_H \in [\hat{\gamma}, \overline{\gamma})$ such that $G(\gamma_H) = 0$, i.e., such that (Rc2) holds. Now apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds $\underline{\gamma}, \gamma_H$ is optimal: (c1) is implied by (Gc1), (c2) by (Rc2) and Lemma 7 part *(i)*, (c2′) vacuously, (c3) vacuously, (c3′) by assumption.

In case *(ii)*, (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. Now apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal. □

Proof of Lemma 3. It holds that $H(\underline{\gamma}) = 0$, and by (13) there exists γ arbitrarily close to $\underline{\gamma}$ such that $H(\gamma) > 0$.[5] Lemma 6 part *(ii)* therefore rules out $\pi^* = \pi_f(\underline{\gamma})$. Moreover, Lemma 6 part *(i)* rules out $\pi^* < \pi_f(\underline{\gamma})$. So there are two possible cases:

(i) $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$.

(ii) $\pi^* \geq \pi_f(\overline{\gamma})$, i.e., $\hat{\gamma} \geq \underline{\gamma}$.

In case *(i)*, first observe that $G(\gamma) \geq 0$ for all γ; this follows from $G(\overline{\gamma}) = 0$ (Lemma 5 part *(a)*(i)), $G'(\overline{\gamma}) \leq 0$ (Lemma 5 part *(b)*(i)), and G convex (Lemma 5 part *(e)*(i)). In particular, $G(\hat{\gamma}) \geq 0$. So by Lemma 5 part *(c)*, it must hold that $H(\hat{\gamma}) \leq 0$. Therefore, continuity of H

[4] Note that the maintained assumption that F has full support on Γ allows for the possibility that $f(\overline{\gamma}) = 0$, and hence that $G'(\overline{\gamma}) = 0$ (Lemma 5 part *(b)*(i)). This is why we appealed directly to (12) to establish that there exists a point $\gamma < \overline{\gamma}$ with $G(\gamma) < 0$. Having established that fact, though, we can actually rule out $f(\overline{\gamma}) = 0$ under the assumptions of the Lemma. In particular, the combination of $G(\overline{\gamma}) = 0$; the existence of a point $\gamma < \overline{\gamma}$ with $G(\gamma) < 0$; and convexity of G, together imply that $G'(\overline{\gamma}) > 0$ and thus that $f(\overline{\gamma}) > 0$.

[5] Analogously to footnote 4 in the proof of Lemma 2, the assumptions of the Lemma imply that $H'(\gamma) > 0$ and thus that $f(\underline{\gamma}) = 0$.

implies that there exists $\gamma_L \in (\underline{\gamma}, \hat{\gamma}]$ such that $H(\gamma_L) = 0$, i.e., such that (Rc3) holds. Now apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds $\gamma_L, \overline{\gamma}$ is optimal: (c1) is implied by (Gc1), (c2) holds vacuously, (c2′) by assumption, (c3) by (Rc3) and Lemma 7 part *(ii)*, (c3′) vacuously.

In case *(ii)*, (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. Now apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal. □

Proof of Lemma 4. It holds that $H(\underline{\gamma}) = G(\overline{\gamma}) = 0$. Moreover, there exists a point γ arbitrarily close to $\underline{\gamma}$ such that $H(\gamma) > 0$ (by (13)), and a point γ arbitrarily close to $\overline{\gamma}$ such that $G(\gamma) < 0$ (by (12)). Consider three cases:

(i) $\pi^* \leq \pi_f(\underline{\gamma})$, i.e., $\hat{\gamma} \leq \underline{\gamma}$.

(ii) $\pi^* \geq \pi_f(\overline{\gamma})$, i.e., $\hat{\gamma} \geq \underline{\gamma}$.

(iii) $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$, in which case $G(\underline{\gamma}) > 0$ and $H(\overline{\gamma}) < 0$ (by Lemma 5 part *(d)*).

In cases *(i)* and *(ii)*, (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. For either of these cases, apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal.

Finally, consider case *(iii)*. By continuity, there must be some γ_L and γ_H in $(\underline{\gamma}, \overline{\gamma})$ such that $H(\gamma_L) = G(\gamma_H) = 0$. By concavity of H, it holds that $H(\gamma) > 0$ for all $\gamma \in (\underline{\gamma}, \gamma_L)$ and $H(\gamma) < 0$ for all $\gamma \in (\gamma_L, \overline{\gamma}]$. Likewise, by convexity of G, it holds that $G(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \gamma_H)$ and $G(\gamma) < 0$ for all $\gamma \in (\gamma_H, \overline{\gamma})$. In other words, for $\gamma < \min\{\gamma_L, \gamma_H\}$, it holds that $H(\gamma) \geq 0$ and $G(\gamma) > 0$, so $H(\gamma) + G(\gamma) > 0$. Hence, from Lemma 5 part *(c)*, for any $\gamma < \min\{\gamma_L, \gamma_H\}$ it holds that $\pi^* > \pi_f(\gamma)$, i.e., that $\hat{\gamma} > \gamma$. Similarly, for $\gamma > \min\{\gamma_L, \gamma_H\}$ it holds that $H(\gamma) + G(\gamma) < 0$, and thus that $\pi^* < \pi_f(\gamma)$, i.e., that $\hat{\gamma} < \gamma$. Putting these observations together, $\hat{\gamma} \in [\min\{\gamma_L, \gamma_H\}, \max\{\gamma_L, \gamma_H\}]$.

Now consider two possibilities in case *(iii)*. The first possibility is that $\gamma_L < \gamma_H$. Then we can apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds γ_L, γ_H is optimal: (c1) is implied by (Gc1), (c2) holds by (Rc2) and Lemma 7 part *(i)*, (c2′) holds vacuously, (c3) holds by (Rc3) and Lemma 7 part *(ii)*, and (c3′) holds vacuously.

The second possibility is that $\gamma_L \geq \gamma_H$. Then $\pi^* \in [\pi_f(\gamma_H), \pi_f(\gamma_L)]$ or, in other words, $\hat{\gamma} \in [\gamma_H, \gamma_L]$. We have that (Rd2) holds because $\hat{\gamma} \geq \gamma_H$, and $G(\gamma) \leq 0$ for any $\gamma \geq \gamma_H$; and (Rd3) holds because $\hat{\gamma} \leq \gamma_L$, and $H(\gamma) \geq 0$ for any $\gamma \leq \gamma_L$. We can now apply Proposition 1 of the current paper to get that the constant allocation at π^* is optimal: (d2) holds by (Rc2) and Lemma 7 part *(i)*, and (d3) holds by (Rd3) and Lemma 7 part *(ii)*. □