

A note on interval delegation

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Abstract In this note we extend the Amador and Bagwell (Econometrica 81:1541–1599, 2013) conditions for confirming the optimality of a proposed interval delegation set to the possibility of degenerate intervals, in which the agent takes the same action at every state. We consider the cases of money burning as well as no money burning. These results allow us to provide new sufficient conditions on utility functions and state distributions to guarantee that some interval—degenerate or non-degenerate—will be optimal.

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JEL Classification D82 · F10

1 Introduction

In a standard delegation problem, a state of the world determines both the principal's and agent's preferences over a one-dimensional action (see, e.g., Holmström 1977). The agent privately observes the state realization. The principal contracts with the agent by giving her a set of actions from which to choose. The problem may also be

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augmented by “money burning,” an auxiliary action that hurts both players. A number of papers provide conditions to guarantee that an interval delegation set is optimal, and that—if money burning is available—no money is burned.¹ Such an interval may take the form of a cap on the actions of an upward-biased agent; a floor on the actions of a downward-biased agent; or a cap together with a floor, for an agent who takes actions that are sometimes too high and sometimes too low.

In many ways the most general sufficient conditions for the optimality of interval delegation, in terms of state distributions and utility functional form assumptions, come from Proposition 1 of Amador and Bagwell (2013). That paper considers problems both with and without money burning, and its utility functions embed those in earlier papers. However, Amador and Bagwell (2013) lack two kinds of results that may be important for readers. First, that paper does not consider degenerate intervals—single-point delegation sets. Second, Amador and Bagwell (2013) find the optimal interval, within the class of intervals, and then give conditions to verify that this proposed interval is optimal within the full class of incentive compatible allocations. But for some applications one might seek sufficient conditions for the optimality of some interval, without first identifying the exact boundaries of that interval.

This note addresses both issues. We first extend the results of Amador and Bagwell (2013) to present conditions for a proposed single point—a degenerate interval—to be an optimal delegation set. We then give sufficiency conditions to guarantee that some interval, degenerate or otherwise, must be optimal. It is enough to require that utility functions are in a certain family, one capturing many of those used in the literature, and that a global regularity condition jointly holds on the utilities and state distributions.

The two issues of degenerate intervals and global sufficiency conditions are inherently linked. The distributional and functional form assumptions which are sufficient to guarantee that some interval is optimal certainly can lead to a degenerate interval. For instance, an agent with a positive bias relative to the principal should be given a cap. But as the agent’s bias grows stronger, the cap is eventually made to be always-binding. So one must address the possibility of degenerate intervals before moving to global sufficiency conditions.

Section 2 presents the result concerning the optimality of degenerate delegation intervals. Section 3 presents the sufficient conditions for the optimality of some interval delegation. Section 4 contains the proof of Proposition 1. An online appendix collects the rest of the proofs.

2 Optimality of degenerate delegation intervals

This paper considers the environment of Amador and Bagwell (2013), maintaining Assumption 1 of that paper throughout. Repeating the key definitions and assumptions, the principal’s utility is given by $w(\gamma, \pi) - t$ and the agent’s by $\gamma\pi + b(\pi) - t$. The

¹ Without money burning, see for instance Melumad and Shibano (1991), Martimort and Semenov (2006), and Alonso and Matouschek (2008). With money burning—captured, in the first two cases, via randomized actions—see Goltsman et al. (2009), Kováč and Mylovanov (2009), and Amador and Bagwell (2013). Ambrus and Egorov (2017) and Amador and Bagwell (2016) study related delegation problems in which the optimal policy may in fact involve some form of money burning incentive.

value $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ represents the state of the world, drawn from a distribution F with continuous density function f and with full support over Γ .² The action is $\pi \in \Pi$, where Π is an interval of the real line with nonempty interior, with $\inf \Pi$ normalized to 0 and $\sup \Pi = \bar{\pi}$ (possibly infinite). The agent’s preferred, or “flexible,” action at state γ is indicated by $\pi_f(\gamma)$. Finally, $t \geq 0$ is the level of money burning.

Assumption 1 The following hold: (i) the function $w : \Gamma \times \Pi \rightarrow \mathbb{R}$ is continuous on $\Gamma \times \Pi$; (ii) for any $\gamma_0 \in \Gamma$, the function $w(\gamma_0, \cdot)$ is concave on Π and twice differentiable on $(0, \bar{\pi})$; (iii) the function $b : \Pi \rightarrow \mathbb{R}$ is strictly concave on Π , and twice differentiable on $(0, \bar{\pi})$; (iv) there exists a twice differentiable function $\pi_f : \Gamma \rightarrow (0, \bar{\pi})$ such that, for all $\gamma_0 \in \Gamma$, $\pi'_f(\gamma_0) > 0$ and $\pi_f(\gamma_0) \in \arg \max_{\pi \in \Pi} \{\gamma_0 \pi + b(\pi)\}$; and (v) the function $w_\pi : \Gamma \times (0, \bar{\pi}) \rightarrow \mathbb{R}$ is continuous on $\Gamma \times (0, \bar{\pi})$, where w_π denotes the derivative of w in its second argument.

The principal chooses an allocation rule $\pi : \Gamma \rightarrow \Pi$ and a transfer rule $t : \Gamma \rightarrow \mathbb{R}$ such that the agent, who privately observes the state, finds it incentive-compatible to report the state truthfully. We denote a pair (π, t) as an *allocation*. We consider two different problems:

- The *problem with money burning* is defined to be:

$$\begin{aligned} \max_{\pi, t} \int_{\Gamma} (w(\gamma, \pi(\gamma)) - t(\gamma)) dF(\gamma) \quad \text{subject to:} \quad & \text{(P)} \\ \gamma \in \arg \max_{\tilde{\gamma} \in \Gamma} \{ \gamma \pi(\tilde{\gamma}) + b(\pi(\tilde{\gamma})) - t(\tilde{\gamma}) \}, \quad & \text{for all } \gamma \in \Gamma \\ t(\gamma) \geq 0, \quad \forall \gamma \in \Gamma & \end{aligned}$$

- The *problem without money burning* is Problem (P) with the additional constraint:

$$t(\gamma) = 0, \quad \forall \gamma \in \Gamma. \tag{1}$$

Just as in Amador and Bagwell (2013), a key parameter is κ , which will be defined as

$$\kappa = \inf_{(\gamma, \pi) \in \Gamma \times \Pi} \left\{ \frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)} \right\}, \quad \text{for the problem without money burning,} \tag{2}$$

$$\kappa = \min \left\{ \inf_{(\gamma, \pi) \in \Gamma \times \Pi} \left\{ \frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)} \right\}, 1 \right\}, \quad \text{for the problem with money burning.} \tag{3}$$

A *constant allocation* is an allocation such that $\pi(\gamma)$ is a constant function, and $t(\gamma) = 0, \forall \gamma \in \Gamma$. The optimal allocation within the class of constant allocations is defined by a value π^* that solves

$$\pi^* \in \arg \max_{\pi} \mathbb{E}_{\tilde{\gamma} \sim F} [w(\tilde{\gamma}, \pi)].$$

We make the following assumption about π^* :

² Amador and Bagwell (2013) impose the stronger assumption that f is strictly positive over Γ . But Proposition 1 of that paper, the only result we will invoke, does not rely on strict positivity of f .

Assumption 2 π^* is in the interior of the action space Π .

The value π^* can, therefore, be characterized as a solution to the first-order condition:

$$\int_{\underline{\gamma}}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) d\tilde{\gamma} = 0. \tag{4}$$

Define $\hat{\gamma} \in \mathbb{R}$ as follows:

$$\hat{\gamma} = -b'(\pi^*). \tag{5}$$

For any $\gamma \in \Gamma$, the first-order condition defining the agent’s preferred action, $\pi_f(\gamma)$, is $\gamma = -b'(\pi_f(\gamma))$. So if $\hat{\gamma} \in \Gamma$, then $\hat{\gamma}$ is the state at which the agent’s ideal point is π^* : $\hat{\gamma} = -b'(\pi_f(\hat{\gamma}))$, implying $\pi_f(\hat{\gamma}) = \pi^*$. However, it may be the case that $\hat{\gamma}$ lies outside the interval Γ , in which case π^* does not correspond to the agent’s ideal point for any state. It holds that (i) $\hat{\gamma} \in \Gamma$ if and only if $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma})]$, and (ii) $\hat{\gamma}$ is strictly increasing in π^* .

Now define two conditions, (d2) and (d3), that will, for degenerate intervals, play the role of (c2) and (c3) from Amador and Bagwell (2013) for nondegenerate intervals:

(d2) If $\hat{\gamma} < \bar{\gamma}$,

$$(\gamma - \hat{\gamma})\kappa \geq \int_{\gamma}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi^*) \frac{f(\tilde{\gamma})}{1 - F(\gamma)} d\tilde{\gamma}, \quad \forall \gamma \geq \hat{\gamma} \text{ in } \Gamma.$$

(d3) If $\hat{\gamma} > \underline{\gamma}$,

$$(\gamma - \hat{\gamma})\kappa \leq \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \pi^*) \frac{f(\tilde{\gamma})}{F(\gamma)} d\tilde{\gamma}, \quad \forall \gamma \leq \hat{\gamma} \text{ in } \Gamma.$$

Amador and Bagwell (2013) had three other conditions that one might need to check: (c1), (c2’), and (c3’). Condition (c1) was relevant only for the interior of a delegation set, and conditions (c2’) and (c3’) applied to the non-binding edges of an interval; these conditions will not be relevant for degenerate intervals.

We now present one of our main results:

Proposition 1 *Optimality of degenerate intervals—sufficiency:*

- (a) (No money burning) *If conditions (d2) and (d3) are satisfied with κ as defined in (2), then the constant allocation π^* solves the problem without money burning, that is, Problem (P) with the additional constraint (1).*
- (b) (Money burning) *If conditions (d2) and (d3) are satisfied with κ as defined in (3), then the constant allocation π^* solves Problem (P).*

One can break the results into three cases. First, we may have $\hat{\gamma} \leq \underline{\gamma}$, which implies $\pi^* \leq \pi_f(\underline{\gamma})$ because b is concave. In this case we need only check (d2). That corresponds to a degenerate cap for an agent with a strong upward bias. In particular, the agent’s ideal point satisfies $\pi_f(\gamma) \geq \pi^*$ for all $\gamma \in \Gamma$. Second, we may have

$\hat{\gamma} \geq \bar{\gamma}$, i.e., $\pi^* \geq \pi_f(\bar{\gamma})$, in which case we need only check (d3). That corresponds to a degenerate floor for an agent with a strong downward bias: $\pi_f(\gamma) \leq \pi^*$ for all $\gamma \in \Gamma$. Finally, $\hat{\gamma}$ may be interior in Γ , i.e., $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma}))$, in which case we need to check both (d2) and (d3).

3 Sufficiency conditions for some interval to be optimal

While (c1) in Amador and Bagwell (2013) was not relevant for checking the optimality of degenerate intervals, some version of (c1) will indeed become relevant for confirming that some interval—degenerate or otherwise—is optimal. Let (Gc1) indicate global (c1), that is, a strengthening of condition (c1) to apply to all $\gamma \in \Gamma$ rather γ in some specified interval.

(Gc1) $\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ is nondecreasing over $\gamma \in \Gamma$.

We now show that (Gc1) combined with a functional form assumption on w implies that some interval is optimal. Specifically, consider the functional form

$$w(\gamma, \pi) = A[b(\pi) + B(\gamma) + C(\gamma)\pi] \quad \text{with } A > 0. \tag{6}$$

When there is no money burning, the value of A can be set to 1 without loss of generality. As shown by Amador and Bagwell (2013), this preference specification encompasses several prominent specifications found in the literature.³

With w of the form (6), the value of κ given by (2), without money burning, is A . The value of κ given by (3), with money burning, is $\min\{A, 1\}$. When we consider money burning below, we make the additional assumption that $A \leq 1$, in which case $\kappa = A$. Moreover, Assumption 1(iii) states that b is strictly concave, and hence w is strictly concave in π (Assumption 1(ii) had only imposed weak concavity). Therefore, $\pi^* \in \arg \max_\pi \mathbb{E}_{\tilde{\gamma} \sim F}[w(\tilde{\gamma}, \pi)]$ will be uniquely defined.

Recall that $w_\pi(\gamma, \pi_f(\gamma))$ gives the agent’s bias at state γ : a negative value indicates that the agent is biased upwards relative to the principal, and a positive value that the agent is biased downwards. In what follows below, we present four lemmas that cover the two-by-two exhaustive cases of an upwards or downwards bias at the lowest state; and an upwards or downwards bias at the highest state.

For the following Lemmas, given $\gamma_L < \gamma_H$ in Γ , an *interval allocation with bounds* γ_L, γ_H refers to a nondegenerate interval delegation set in which the agent can select any action in the range $[\pi_f(\gamma_L), \pi_f(\gamma_H)]$, and there is no money burning:

Lemma 1 (Unconstrained interval) *Let w be of the form (6). Suppose that (Gc1) holds with $\kappa = A$. Suppose that $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$ and $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) \geq 0$. Then the*

³ For example, the preferences assumed by Alonso and Matouschek (2008)—which does not allow for money burning—are a special case. Restating that conclusion, the primary model of Alonso and Matouschek (2008) proposes an agent utility function of $v_A(\pi - \pi_f(\gamma), \gamma)$, for v_A single-peaked and symmetric about 0 in its first term; a principal utility of $-r(\gamma)(\pi - \pi_P(\gamma))^2$, for r everywhere positive; and no money burning. Without loss, they can transform the problem so that $\pi_f(\gamma) = \gamma$ and $r(\gamma) = 1$. Indeed, the agent’s behavior given any delegation set is as if her utility function were $\gamma\pi + b(\pi)$ for $b(\pi) = -\pi^2/2$. The principal’s utility can then be written as $w(\gamma, \pi)$ as in (6) with $A = 1$, $C(\gamma) = \pi_P(\gamma)$, and $B(\gamma) = -(\pi_P(\gamma))^2/2$.

solution to Problem (P) with additional constraint (1); or, under the further assumption that $A \leq 1$, the solution to Problem (P); is given by the interval allocation with bounds $\underline{\gamma}, \bar{\gamma}$. Moreover, it holds that $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma})]$.

Lemma 2 (Cap) *Let w be of the form (6). Suppose that (Gc1) holds with $\kappa = A$. Suppose that $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$ and $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) < 0$. Then the solution to Problem (P) with additional constraint (1); or, under the further assumption that $A \leq 1$, the solution to Problem (P); is as follows:*

- (i) *If $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma}))$: The interval allocation with bounds $\underline{\gamma}, \gamma_H$ is optimal, for some $\gamma_H \in [\hat{\gamma}, \bar{\gamma})$.*
- (ii) *If $\pi^* \leq \pi_f(\underline{\gamma})$: The constant allocation π^* is optimal.*

It cannot hold that $\pi^ \geq \pi_f(\bar{\gamma})$.*

Lemma 3 (Floor) *Let w be of the form (6). Suppose that (Gc1) holds with $\kappa = A$. Suppose that $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$ and $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) \geq 0$. Then the solution to Problem (P) with additional constraint (1); or, under the further assumption that $A \leq 1$, the solution to Problem (P); is as follows:*

- (i) *If $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma}))$: The interval allocation with bounds $\gamma_L, \bar{\gamma}$ is optimal, for some $\gamma_L \in (\underline{\gamma}, \hat{\gamma}]$.*
- (ii) *If $\pi^* \geq \pi_f(\bar{\gamma})$: The constant allocation π^* is optimal.*

It cannot hold that $\pi^ \leq \pi_f(\underline{\gamma})$.*

Lemma 4 (Cap and Floor) *Let w be of the form (6). Suppose that (Gc1) holds with $\kappa = A$. Suppose that $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$ and $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) < 0$. Then the solution to Problem (P) with additional constraint (1); or, under the further assumption that $A \leq 1$, the solution to Problem (P); is given by one of the following cases:*

- (i) *The interval allocation with bounds γ_L, γ_H is optimal, for some γ_L, γ_H satisfying $\underline{\gamma} < \gamma_L < \gamma_H < \bar{\gamma}$. In this case it must hold that $\pi^* \in [\pi_f(\gamma_L), \pi_f(\gamma_H)]$.*
- (ii) *The constant allocation π^* is optimal.*

The proofs of these lemmas build off of an argument in Amador and Bagwell (2016, Lemma 1). Under the functional form assumption (6), rather than checking (c2), (c3), (d2) and (d3) over an interval, condition (Gc1) allows us to check the conditions just at the boundary point. As we explain in the proofs, a related argument is developed by Alonso and Matouschek (2008). For the payoffs that they consider, condition (Gc1) implies the convexity or concavity of their “forward bias” and “backward bias” expressions (which play an important role in their argument).

Lemma 1 gives conditions for the optimality of an unconstrained delegation interval, with no binding cap or floor. This occurs when the agent is “more moderate” than the principal, being biased towards higher actions at the lowest state and lower actions at the highest state. Lemma 2 gives conditions for a cap, possibly binding for only some states (part (i)) and possibly degenerate and binding for all states (part (ii)). This occurs when the agent is biased upwards at the lowest and highest states. When the cap is not always binding (part (iii)), the cap is above the principal’s ex ante optimal action π^* .

Lemma 3 gives similar conditions for a floor, when the agent is biased downwards at the lowest and highest states.

Lemma 4 gives conditions for interval delegation in the remaining case, in which the agent is biased downwards at the lowest state and upwards at the highest state. This can be thought of as an “extreme-biased” agent, although it is also consistent with simple misalignment of preferences: the principal’s ideal point may be falling over some range of states as the agent’s ideal point rises. Part (i) describes the case of a floor combined with a cap, where the floor is strictly below the cap. Part (ii) actually combines multiple cases. It may be that $\pi^* \leq \pi_f(\underline{\gamma})$, in which case the contract is an always-binding cap. It may be that $\pi^* \geq \pi_f(\overline{\gamma})$, in which case the contract is an always-binding floor. Or it may be that π^* is in between $\pi_f(\underline{\gamma})$ and $\pi_f(\overline{\gamma})$, in which case the contract can be thought of as a cap and a floor set at the same point.

Putting together these four lemmas:

Proposition 2 *Let w be of the form (6). Suppose that (Gc1) holds with $\kappa = A$.*

- (a) *(No money burning) Then the solution to Problem (P) with additional constraint (1) is given by some interval delegation set, possibly degenerate.*
- (b) *(Money burning) Suppose also that $A \leq 1$. Then the solution to Problem (P) is given by some interval delegation set, possibly degenerate.*

4 Proof of Proposition 1

We follow closely the proof of Proposition 1 in Amador and Bagwell (2013). The main difference is the proposed Lagrange multipliers, as in this case we are allowing for the possibility that the optimal allocation involves no flexibility. Once these multipliers have been found, the steps of the proof are identical. See Amador and Bagwell (2013) for a more detailed discussion of these steps. We will prove each of the parts of this Proposition separately.

Proof of part (a) of Proposition 1: without money burning

We first write the incentive constraints in their usual monotonicity restriction plus an integral form. Just as in Amador and Bagwell (2013), we write the integral form as two inequalities:

$$\int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma})d\tilde{\gamma} + \underline{U} - \gamma\pi(\gamma) - b(\pi(\gamma)) \leq 0, \quad \text{for all } \gamma \in \Gamma, \tag{7}$$

$$- \int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma})d\tilde{\gamma} - \underline{U} + \gamma\pi(\gamma) + b(\pi(\gamma)) \leq 0, \quad \text{for all } \gamma \in \Gamma, \tag{8}$$

where $\underline{U} \equiv \underline{\gamma}\pi(\underline{\gamma}) + b(\pi(\underline{\gamma}))$.

The problem is then to choose a function $\pi \in \Phi$ so as to maximize

$$\max_{\pi : \Gamma \rightarrow \Pi} \int w(\gamma, \pi(\gamma))dF(\gamma)$$

subject to (7) and (8) and where the choice set incorporates the monotonicity restriction: $\Phi \equiv \{\pi | \pi : \Gamma \rightarrow \Pi \text{ and } \pi \text{ non-decreasing}\}$.

We then assign cumulative Lagrange multiplier functions Λ_1 and Λ_2 to constraints (7) and (8), respectively, and write the Lagrangian for the problem and after integrating by parts we obtain

$$\begin{aligned} \mathcal{L}(\pi | \Lambda) &= \int_{\Gamma} [w(\gamma, \pi(\gamma))f(\gamma) - (\Lambda(\bar{\gamma}) - \Lambda(\gamma))\pi(\gamma)]d\gamma \\ &\quad + \int_{\Gamma} (\gamma\pi(\gamma) + b(\pi(\gamma)))d\Lambda(\gamma) - \underline{U}(\Lambda(\bar{\gamma}) - \Lambda(\underline{\gamma})), \end{aligned}$$

where $\Lambda(\gamma) \equiv \Lambda_1(\gamma) - \Lambda_2(\gamma)$.

A proposed multiplier Let us propose some non-decreasing multipliers Λ_1 and Λ_2 so that their difference, Λ , satisfies:

$$\Lambda(\gamma) = \begin{cases} 1 & \gamma = \bar{\gamma} \\ 1 + \kappa(1 - F(\gamma)) & \gamma \in (\gamma^*, \bar{\gamma}) \\ 1 + \kappa(1 - F(\gamma^*)) & \gamma = \gamma^* \\ 1 - \kappa F(\gamma) & \gamma \in (\underline{\gamma}, \gamma^*) \\ 1 & \gamma = \underline{\gamma} \end{cases} \quad \text{and } \gamma^* \in (\underline{\gamma}, \bar{\gamma}),$$

where κ is given by (2) and $\gamma^* = \min \{ \max\{\hat{\gamma}, \underline{\gamma}\}, \bar{\gamma} \} \in \Gamma$ where $\hat{\gamma}$ is given by (5).⁴ Note that Λ is well defined even when $\hat{\gamma}$ lies outside $[\underline{\gamma}, \bar{\gamma}]$.

Below we show that $\kappa F(\gamma) + \Lambda(\gamma) \equiv R(\gamma)$ is non-decreasing; hence, it follows that $\Lambda(\gamma)$ can indeed be written as the difference of two non-decreasing functions, $R(\gamma) - \kappa F(\gamma)$.

Concavity of the Lagrangian The next step is to check that the Lagrangian is concave when evaluated at the proposed multipliers. Towards this goal, we first note that the jump at γ^* in Λ equals κ , and thus is non-negative. Using that $\Lambda(\bar{\gamma}) = \Lambda(\underline{\gamma}) = 1$, we can write the Lagrangian as follows:

$$\begin{aligned} \mathcal{L}(\pi | \Lambda) &= \int_{\Gamma} [w(\gamma, \pi(\gamma)) - \kappa(\gamma\pi(\gamma) + b(\pi(\gamma)))]f(\gamma)d\gamma - \int_{\Gamma} (1 - \Lambda(\gamma))\pi(\gamma)d\gamma \\ &\quad + \int_{\Gamma} (\gamma\pi(\gamma) + b(\pi(\gamma)))d(\kappa F(\gamma) + \Lambda(\gamma)). \end{aligned} \tag{9}$$

The concavity of $w(\gamma, \pi(\gamma)) - \kappa b(\pi(\gamma))$ in $\pi(\gamma)$ follows from the definition of κ . The fact that its jump at γ^* is non-negative implies that $\kappa F(\gamma) + \Lambda(\gamma)$ is non-decreasing.

Maximizing the Lagrangian We now proceed to show that the proposed allocation π^* maximizes the Lagrangian.

⁴ Here we use the convention that the open interval (x, x) corresponds to the empty set, \emptyset .

As in Amador and Bagwell (2013), we extend b and w to the entire positive ray of the real line. Denoting $\hat{\Phi} = \{\pi | \pi : \Gamma \rightarrow \mathbb{R}_+$ and π non-decreasing}, we can say that if

$$\begin{aligned} \partial \mathcal{L}(\pi^*; \pi^* | \Lambda) &= 0, \\ \partial \mathcal{L}(\pi^*; \mathbf{x} | \Lambda) &\leq 0; \quad \text{for all } \mathbf{x} \in \hat{\Phi}, \end{aligned}$$

then π^* maximizes the Lagrangian \mathcal{L} .

For our problem, taking the Gateaux differential in direction $\mathbf{x} \in \hat{\Phi}$. we get that:⁵

$$\begin{aligned} \partial \mathcal{L}(\pi^*; \mathbf{x} | \Lambda) &= \int_{\Gamma} [w_{\pi}(\gamma, \pi^*)f(\gamma) - (1 - \Lambda(\gamma))] \mathbf{x}(\gamma) d\gamma \\ &\quad + \int_{\underline{\gamma}}^{\bar{\gamma}} (\gamma - \hat{\gamma}) \mathbf{x}(\gamma) d\Lambda(\gamma) \end{aligned} \tag{10}$$

which can be rewritten as

$$\begin{aligned} \partial \mathcal{L}(\pi^*; \mathbf{x} | \Lambda) &= \int_{\underline{\gamma}}^{\gamma^*} [w_{\pi}(\gamma, \pi^*)f(\gamma) - \kappa F(\gamma) - \kappa(\gamma - \hat{\gamma})f(\gamma)] \mathbf{x}(\gamma) d\gamma \\ &\quad + \int_{\gamma^*}^{\bar{\gamma}} [w_{\pi}(\gamma, \pi^*)f(\gamma) + \kappa(1 - F(\gamma)) - \kappa(\gamma - \hat{\gamma})f(\gamma)] \mathbf{x}(\gamma) d\gamma \\ &\quad + \kappa(\gamma^* - \hat{\gamma})\mathbf{x}(\gamma^*) \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \partial \mathcal{L}(\pi^*; \mathbf{x} | \Lambda) &= \mathbf{x}(\gamma^*) \left\{ \int_{\underline{\gamma}}^{\gamma^*} [w_{\pi}(\gamma, \pi^*)f(\gamma) - \kappa F(\gamma) - \kappa(\gamma - \hat{\gamma})f(\gamma)] d\gamma \right. \\ &\quad \left. + \int_{\gamma^*}^{\bar{\gamma}} [w_{\pi}(\gamma, \pi^*)f(\gamma) + \kappa(1 - F(\gamma)) - \kappa(\gamma - \hat{\gamma})f(\gamma)] d\gamma + \kappa(\gamma^* - \hat{\gamma}) \right\} \\ &\quad - \int_{\underline{\gamma}}^{\gamma^*} \left[\int_{\underline{\gamma}}^{\gamma} [w_{\pi}(\tilde{\gamma}, \pi^*)f(\tilde{\gamma}) - \kappa F(\tilde{\gamma}) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} \right] d\mathbf{x}(\gamma) \\ &\quad + \int_{\gamma^*}^{\bar{\gamma}} \left[\int_{\gamma}^{\bar{\gamma}} [w_{\pi}(\tilde{\gamma}, \pi^*)f(\tilde{\gamma}) + \kappa(1 - F(\tilde{\gamma})) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} \right] d\mathbf{x}(\gamma). \end{aligned} \tag{11}$$

We require that this differential be non-positive for all non-decreasing \mathbf{x} and zero when evaluated at $\mathbf{x} = \pi^*$. Note that if $\mathbf{x} = \pi^*$, then $d\mathbf{x}(\gamma) = 0$. So we have that

⁵ Existence of the Gateaux differentials and the ability to integrate by parts follows from identical arguments to those in footnotes 52 and 53 of Amador and Bagwell (2013).

$$\begin{aligned} \partial\mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{\pi}^*|\Lambda) &= \boldsymbol{\pi}^* \left\{ \int_{\underline{\gamma}}^{\gamma^*} [w_{\pi}(\gamma, \boldsymbol{\pi}^*)f(\gamma) - \kappa F(\gamma) - \kappa(\gamma - \hat{\gamma})f(\gamma)] d\gamma \right. \\ &\quad + \int_{\gamma^*}^{\bar{\gamma}} [w_{\pi}(\gamma, \boldsymbol{\pi}^*)f(\gamma) + \kappa(1 - F(\gamma)) - \kappa(\gamma - \hat{\gamma})f(\gamma)] d\gamma \\ &\quad \left. + \kappa(\gamma^* - \hat{\gamma}) \right\} \\ &= \kappa\boldsymbol{\pi}^* \left[\int_{\underline{\gamma}}^{\bar{\gamma}} [-F(\gamma) - \gamma f(\gamma)] d\gamma + \bar{\gamma} \right] = 0, \end{aligned}$$

where the last equality follows from (4) and the identity $\int_a^b \gamma f(\gamma) d\gamma = bF(b) - aF(a) - \int_a^b F(\gamma) d\gamma$.

To guarantee that $\partial\mathcal{L}(\boldsymbol{\pi}^*; \mathbf{x}|\Lambda) \leq 0$ for all $\mathbf{x} \in \hat{\Phi}$, we need that

$$\begin{aligned} \int_{\underline{\gamma}}^{\gamma} [w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) - \kappa F(\tilde{\gamma}) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} &\geq 0 \quad \text{for all } \gamma \in [\underline{\gamma}, \gamma^*] \\ \int_{\gamma}^{\bar{\gamma}} [w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) + \kappa(1 - F(\tilde{\gamma})) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} &\leq 0 \quad \text{for all } \gamma \in (\gamma^*, \bar{\gamma}], \end{aligned}$$

where this follows by noticing $\partial\mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{\pi}^*|\Lambda) = 0$ implies that the term in curly brackets in Eq. (11) equals zero.

These conditions in turn are implied by (d2) and (d3). To see, note that

$$\begin{aligned} &\int_{\underline{\gamma}}^{\gamma} [w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) - \kappa F(\tilde{\gamma}) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} \\ &= \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) d\tilde{\gamma} - \kappa(\gamma - \hat{\gamma})F(\gamma) \end{aligned}$$

And thus (d3), together with the definition of γ^* , implies that

$$\int_{\underline{\gamma}}^{\gamma} [w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) - \kappa F(\tilde{\gamma}) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} \geq 0 \quad \text{for all } \gamma \in [\underline{\gamma}, \gamma^*]$$

A similar argument shows that (d2) implies

$$\int_{\gamma}^{\bar{\gamma}} [w_{\pi}(\tilde{\gamma}, \boldsymbol{\pi}^*)f(\tilde{\gamma}) + \kappa(1 - F(\tilde{\gamma})) - \kappa(\tilde{\gamma} - \hat{\gamma})f(\tilde{\gamma})] d\tilde{\gamma} \leq 0 \quad \text{for all } \gamma \in (\gamma^*, \bar{\gamma}]$$

Hence, using concavity of Lagrangian plus Lemma A.2 in Amador et al. (2006), we have shown that the proposed allocation $\boldsymbol{\pi}^*$ maximizes the Lagrangian given the multipliers.

Applying Luenberger's sufficiency theorem Just as in Amador and Bagwell (2013), we then apply Theorem 1 in their appendix to show that π^* is an optimal solution of the original problem.

Proof of part (b) of Proposition 1: with money burning

The proof of part (b) follows the same steps as in Amador and Bagwell (2013), but this time with the multiplier given by

$$\tilde{\Lambda}(\gamma) = \begin{cases} 1 & \gamma = \bar{\gamma} \\ (1 - \kappa)F(\gamma) + \kappa & \gamma \in (\gamma^*, \bar{\gamma}) \\ (1 - \kappa)F(\gamma^*) + \kappa & \gamma = \gamma^* \\ (1 - \kappa)F(\gamma) & \gamma \in (\underline{\gamma}, \gamma^*) \\ 0 & \gamma = \underline{\gamma} \end{cases} \quad \text{and} \quad \gamma^* \in (\underline{\gamma}, \bar{\gamma}),$$

where κ is given by definition (3), and $\gamma^* = \min\{\max\{\hat{\gamma}, \underline{\gamma}\}, \bar{\gamma}\} \in \Gamma$ where $\hat{\gamma}$ is given by (5).

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Online appendix to “A Note on Interval Delegation”

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This appendix collects the proofs of Lemmas 1 through 4.

A Proof of Lemmas 1 - 4

In Proposition 1 of [Amador and Bagwell \(2013\)](#), conditions (c1), (c2), (c2'), (c3), and (c3') all must hold for some specified $\gamma_L < \gamma_H$ in Γ in order for the (non-degenerate) interval allocation with bounds γ_L, γ_H to be optimal. Condition (c1) will be implied by our condition (Gc1). Let us restate the other conditions:

(c2) If $\gamma_H < \bar{\gamma}$,

$$(\gamma - \gamma_H)\kappa \geq \int_{\gamma}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_H)) \frac{f(\tilde{\gamma})}{1 - F(\tilde{\gamma})} d\tilde{\gamma}, \quad \forall \gamma \in [\gamma_H, \bar{\gamma}]$$

with equality at γ_H .

(c2') If $\gamma_H = \bar{\gamma}$, $w_{\pi}(\bar{\gamma}, \pi_f(\bar{\gamma})) \geq 0$.

(c3) If $\gamma_L > \underline{\gamma}$,

$$(\gamma - \gamma_L)\kappa \leq \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_L)) \frac{f(\tilde{\gamma})}{F(\tilde{\gamma})} d\tilde{\gamma}, \quad \forall \gamma \in [\underline{\gamma}, \gamma_L]$$

with equality at γ_L .

(c3') If $\gamma_L = \underline{\gamma}$, $w_{\pi}(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$.

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Now we proceed with new definitions. First, in the case that $\hat{\gamma} \notin \Gamma$, extend π_f from the domain Γ to the domain $\Gamma \cup \{\hat{\gamma}\}$ by letting $\pi_f(\hat{\gamma}) = \pi^*$. Recall the discussion after Equation (5), that $\gamma = -b'(\pi_f(\gamma))$ for all $\gamma \in \Gamma \cup \{\hat{\gamma}\}$ and that $\pi_f(\hat{\gamma}) = \pi^*$ by construction when $\hat{\gamma} \in \Gamma$. It is the case that π_f is strictly increasing over this extended domain, and so the sign of $\gamma - \hat{\gamma}$ is the same as the sign of $\pi_f(\gamma) - \pi_f(\hat{\gamma})$.

Next, define the functions $G : \Gamma \rightarrow \mathbb{R}$ and $H : \Gamma \rightarrow \mathbb{R}$ as follows.

$$G(\gamma) \equiv \int_{\gamma}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) f(\tilde{\gamma}) d\tilde{\gamma} \quad (12)$$

$$H(\gamma) \equiv \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) f(\tilde{\gamma}) d\tilde{\gamma}, \quad (13)$$

The following lemma summarizes some properties of G and H , which will be useful for later reference.

Lemma 5. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds. Then the functions G and H are continuous, with*

- (a) (i) $G(\bar{\gamma}) = 0$, and (ii) $H(\underline{\gamma}) = 0$.
- (b) (i) $G'(\bar{\gamma}) = -w_{\pi}(\bar{\gamma}, \pi_f(\bar{\gamma}))f(\bar{\gamma})$, and (ii) $H'(\underline{\gamma}) = w_{\pi}(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma})$.¹
- (c) $H(\gamma) + G(\gamma)$ has the same sign as $\pi^* - \pi_f(\gamma)$.
- (d) (i) $G(\underline{\gamma})$ has the same sign as $\pi^* - \pi_f(\underline{\gamma})$, and (ii) $H(\bar{\gamma})$ has the same sign as $\pi^* - \pi_f(\bar{\gamma})$.
- (e) (i) G is weakly convex, and (ii) H is weakly concave.
- (f) For any $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma \cup \{\hat{\gamma}\}$,

$$G(\gamma) = \int_{\gamma}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_0)) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma - \gamma_0)\kappa(1 - F(\gamma)) \quad (14)$$

$$H(\gamma) = \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_0)) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma - \gamma_0)\kappa F(\gamma). \quad (15)$$

Proof of Lemma 5. Continuity as well as parts (a) and (b) are straightforward.

To show parts (c) and (d), note that strict concavity of w in its second term implies that $\int_{\underline{\gamma}}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi) f(\tilde{\gamma}) d\tilde{\gamma}$ is strictly decreasing in π . The action π^* satisfies the first-order condition (4), that $\int_{\underline{\gamma}}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) d\tilde{\gamma} = 0$. Therefore, $\int_{\underline{\gamma}}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi) f(\tilde{\gamma}) d\tilde{\gamma}$ has the sign of $\pi^* - \pi$. Part

¹Take $G'(\bar{\gamma})$ as the left-derivative and $H'(\underline{\gamma})$ as the right-derivative at these points.

(c) then follows from observing that $H(\gamma) + G(\gamma) = \int_{\underline{\gamma}}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) f(\tilde{\gamma}) d\tilde{\gamma}$. Part (d) (i) and (ii) are special cases of part (c), respectively applying part (a) (ii) and (i).

To show part (e), observe that for w of the form (6) it holds that

$$w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) = A[b'(\pi_f(\gamma)) + C(\tilde{\gamma})] = A[C(\tilde{\gamma}) - \gamma], \quad (16)$$

where the last equality follows from the FOC $b'(\pi_f(\gamma)) + \gamma = 0$. Plugging $w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) = A[C(\tilde{\gamma}) - \gamma]$ into $G(\gamma)$ as defined in (12) and taking the derivative yields

$$\begin{aligned} G'(\gamma) &= -A[C(\gamma) - \gamma]f(\gamma) - \int_{\gamma}^{\bar{\gamma}} Af(\tilde{\gamma})d\tilde{\gamma} \\ &= -w_{\pi}(\gamma, \pi_f(\gamma))f(\gamma) - A(1 - F(\gamma)) \\ &= [AF(\gamma) - w_{\pi}(\gamma, \pi_f(\gamma))f(\gamma)] - A \end{aligned}$$

which is nondecreasing by (Gc1). Therefore G is weakly convex. Similarly, plugging $w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) = A[C(\tilde{\gamma}) - \gamma]$ into $H(\gamma)$ as defined in (13) and taking the derivative yields

$$\begin{aligned} H'(\gamma) &= A[C(\gamma) - \gamma]f(\gamma) - \int_{\underline{\gamma}}^{\gamma} Af(\tilde{\gamma})d\tilde{\gamma} \\ &= w_{\pi}(\gamma, \pi_f(\gamma))f(\gamma) - AF(\gamma) \end{aligned}$$

which is nonincreasing by (Gc1). Therefore H is weakly concave.

To show part (f), first note from (16) that $w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) = A[C(\tilde{\gamma}) - \gamma]$ and that $w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_0)) = A[C(\tilde{\gamma}) - \gamma_0]$ for any γ and γ_0 . Combining these two equations, $w_{\pi}(\tilde{\gamma}, \pi_f(\gamma)) = w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_0)) - A[\gamma - \gamma_0]$. Substituting this identity into (12) and (13) and integrating out $A[\gamma - \gamma_0]$ yields (14) and (15), for $\kappa = A$. \square

The functions G and H will essentially take the place of the “forward” and “backward” biases from [Alonso and Matouschek \(2008\)](#). Specifically, after flipping the sign of both functions, G generalizes the forward bias, and H generalizes the backward bias. In [Alonso and Matouschek \(2008\)](#), the convexity of the backward bias, and the corresponding concavity of the forward bias, are important for establishing optimality of interval delegation. After the sign changes, that translates to the convexity of G and the concavity of H in Lemma (5) part (e).²

Putting together parts (a), (b), (d), and (e) of Lemma 5 yields the following result:

Lemma 6. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds.*

²Letting $\pi_P(\gamma)$ indicate the principal’s preferred action at state γ , and taking the agent’s preferred action $\pi_f(\gamma)$ to be linear in γ , [Alonso and Matouschek \(2008\)](#) define the forward bias $S(\gamma)$ and backward bias $T(\gamma)$

- (i) If $\pi^* < \pi_f(\underline{\gamma})$, then $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) < 0$.
- (ii) If $\pi^* = \pi_f(\underline{\gamma})$, then $H(\gamma) < 0$ for all $\gamma > \underline{\gamma}$.
- (iii) If $\pi^* = \pi_f(\bar{\gamma})$, then $G(\gamma) > 0$ for all $\gamma < \bar{\gamma}$.
- (iv) If $\pi^* > \pi_f(\bar{\gamma})$, then $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$.

Proof of Lemma 6. (i) If $\pi^* < \pi_f(\underline{\gamma})$ then $G(\underline{\gamma}) < 0$ (Lemma 5 part (d)(i)); and $G(\bar{\gamma}) = 0$ (Lemma 5 part (a)(i)). Therefore by convexity of G (Lemma 5 part (e)(i)), it must be that $G'(\bar{\gamma}) > 0$. Hence, by Lemma 5 part (b)(i), $w_\pi(\bar{\gamma}, \pi_f(\bar{\gamma})) < 0$.

(ii) If $\pi^* = \pi_f(\underline{\gamma})$ then $G(\underline{\gamma}) = 0$ (Lemma 5 part (d)(i)) and $G(\bar{\gamma}) = 0$ (Lemma 5 part (a)(i)). Therefore, by convexity of G (Lemma 5 part (e)(i)), it holds that $G(\gamma) = 0$ for all γ . The result then follows from Lemma 5 part (c).

(iii) If $\pi^* > \pi_f(\bar{\gamma})$ then $H(\bar{\gamma}) > 0$ (Lemma 5 part (d)(ii)); and $H(\underline{\gamma}) = 0$ (Lemma 5 part (a)(ii)). Therefore by concavity of H (Lemma 5 part (e)(ii)), it must be that $H'(\underline{\gamma}) > 0$. Hence, by Lemma 5 part (b)(ii), $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$.

(iv) If $\pi^* = \pi_f(\bar{\gamma})$ then $H(\bar{\gamma}) = 0$ (Lemma 5 part (d)(ii)) and $H(\underline{\gamma}) = 0$ (Lemma 5 part (a)(ii)). Therefore, by concavity of H (Lemma 5 part (e)(ii)), it holds that $H(\gamma) = 0$ for all γ . The result then follows from Lemma 5 part (c). □

We now proceed to prove Lemma 1 from the main text.

as

$$S(\gamma) \equiv (1 - F(\gamma))\pi_f(\gamma) - \int_{\gamma}^{\bar{\gamma}} \pi_P(\tilde{\gamma})f(\tilde{\gamma})d\tilde{\gamma}$$

$$T(\gamma) \equiv F(\gamma)\pi_f(\gamma) - \int_{\underline{\gamma}}^{\gamma} \pi_P(\tilde{\gamma})f(\tilde{\gamma})d\tilde{\gamma}.$$

While [Alonso and Matouschek \(2008\)](#) focus on the convexity of T , linearity of π_f makes that equivalent to the concavity of S .

Under the functional form (6), we can plug (16) into (12) and (13) to get that

$$G(\gamma) = \int_{\gamma}^{\bar{\gamma}} A(C(\tilde{\gamma}) - \gamma)f(\tilde{\gamma})d\tilde{\gamma} = -A \left((1 - F(\gamma))\gamma - \int_{\gamma}^{\bar{\gamma}} C(\tilde{\gamma})f(\tilde{\gamma})d\tilde{\gamma} \right)$$

$$H(\gamma) = \int_{\underline{\gamma}}^{\gamma} A(C(\tilde{\gamma}) - \gamma)f(\tilde{\gamma})d\tilde{\gamma} = -A \left(F(\gamma)\gamma - \int_{\underline{\gamma}}^{\gamma} C(\tilde{\gamma})f(\tilde{\gamma})d\tilde{\gamma} \right).$$

As described in footnote 3 in the main text, the problem of [Alonso and Matouschek \(2008\)](#) can be transformed to one with utility of the form (6) in which $\pi_f(\gamma) = \gamma$, $A = 1$, and $C(\gamma) = \pi_P(\gamma)$. The above expressions then imply that $G(\gamma) = -S(\gamma)$ and $H(\gamma) = -T(\gamma)$.

Proof of Lemma 1. The optimal allocation follows as an immediate application of Proposition 1 of Amador and Bagwell (2013) with $\gamma_L = \underline{\gamma}$ and $\gamma_H = \bar{\gamma}$, noting that (c1) is implied by (Gc1); (c2) holds vacuously; (c2') holds by assumption; (c3) holds vacuously; and (c3') holds by assumption. The fact that $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma})]$ follows from Lemma 6 parts (i) and (iv). \square

To prove Lemmas 2 through 4, let us now write “relaxed” versions of (c2), (c3), (d2), and (d3). The relaxed version of (c2) just confirms that the condition holds with equality at $\gamma = \gamma_H$, rather than additionally checking weak inequality at $\gamma > \gamma_H$; call this (Rc2). Likewise, call (Rc3) the relaxation of (c3) to just hold with equality at $\gamma = \gamma_L$. Call (Rd2) and (Rd3) the relaxations of (d2) and (d3) in which the relevant inequalities hold only at $\hat{\gamma}$, and only when $\hat{\gamma}$ is on the interior of Γ . These new conditions can be written in terms of G and H .

(Rc2) If $\gamma_H < \bar{\gamma}$ in Γ , $G(\gamma_H) = 0$.

(Rc3) If $\gamma_L > \underline{\gamma}$ in Γ , $H(\gamma_L) = 0$.

(Rd2) If $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$, $G(\hat{\gamma}) \leq 0$.

(Rd3) If $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$, $H(\hat{\gamma}) \geq 0$.

In fact, each of the relaxed conditions will be sufficient to imply the original conditions.

Lemma 7. *Let w be of the form (6), let $\kappa = A$, and suppose that (Gc1) holds.*

(i) *Fixing $\gamma_H < \bar{\gamma}$ in Γ , (Rc2) implies (c2).*

(ii) *Fixing $\gamma_L > \underline{\gamma}$ in Γ , (Rc3) implies (c3).*

(iii) *(Rd2) implies (d2).*

(iv) *(Rd3) implies (d3).*

Part (i) is a restatement of a result in Lemma 1 of Amador and Bagwell (2016). The proof follows exactly as in that Lemma, relying on the convexity of G and its expression as (14).³ The other parts extend similar arguments from the case of a cap at state γ_H to the cases of a floor at state γ_L , and to floors or caps at action π^* .

³More precisely, the proof of Lemma 1 of Amador and Bagwell (2016) defines a function $G(\gamma)$ directly as the expression for $G(\gamma)$ in (14), with $\gamma_0 = \gamma_H$. They show that (Gc1) implies convexity of this function over Γ , as in Lemma 5 part (e)(i) of the current paper, and that convexity implies the result.

Proof of Lemma 7. (i) Fix $\gamma_H < \bar{\gamma}$ and suppose that (Rc2) holds. Applying Equation (14) with $\gamma_0 = \gamma_H$, condition (c2) is equivalent to

$$G(\gamma) \leq 0 \text{ for } \gamma \geq \gamma_H \text{ in } \Gamma, \text{ with } G(\gamma_H) = 0.$$

It holds that $G(\bar{\gamma}) = 0$, and that $G(\gamma_H) = 0$ under (Rc2). So (c2) follows from convexity of G (Lemma 5 part (e)(i)).

(ii) Fix $\gamma_L > \underline{\gamma}$ and suppose that (Rc3) holds. Applying Equation (15) with $\gamma_0 = \gamma_L$, condition (c3) is equivalent to

$$H(\gamma) \geq 0 \text{ for } \gamma \leq \gamma_L \text{ in } \Gamma, \text{ with } H(\gamma_L) = 0.$$

It holds that $H(\underline{\gamma}) = 0$, and that $H(\gamma_L) = 0$ under (Rc3). So (c3) follows from concavity of H (Lemma 5 part (e)(ii)).

(iii) If $\hat{\gamma} \geq \bar{\gamma}$, then (d2) holds vacuously. If $\hat{\gamma} < \bar{\gamma}$, applying Equation (14) with $\gamma_0 = \hat{\gamma}$ shows that condition (d2) is equivalent to

$$G(\gamma) \leq 0 \text{ for } \gamma \geq \hat{\gamma} \text{ in } \Gamma.$$

It holds that $G(\bar{\gamma}) = 0$. So if $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$, then by convexity of G , (d2) is implied by $G(\hat{\gamma}) \leq 0$, which is the condition (Rd2). On the other hand, if $\hat{\gamma} \leq \underline{\gamma}$ (i.e., $\pi^* \leq \pi_f(\underline{\gamma})$), then by convexity of G , (d2) is implied by $G(\underline{\gamma}) \leq 0$; and $G(\underline{\gamma}) \leq 0$ holds by Lemma 5 part (d)(i).

(iv) If $\hat{\gamma} \leq \underline{\gamma}$, then (d3) holds vacuously. If $\hat{\gamma} > \underline{\gamma}$, applying Equation (15) with $\gamma_0 = \hat{\gamma}$, condition (d3) is equivalent to

$$H(\gamma) \geq 0 \text{ for } \gamma \leq \hat{\gamma} \text{ in } \Gamma.$$

It holds that $H(\underline{\gamma}) = 0$. So if $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$, then by concavity of H , (d3) is implied by $H(\hat{\gamma}) \geq 0$, which is the condition (Rd3). On the other hand, if $\hat{\gamma} \geq \bar{\gamma}$ (i.e., $\pi^* \geq \pi_f(\bar{\gamma})$), then by concavity of H , (d3) is implied by $H(\bar{\gamma}) \geq 0$; and $H(\bar{\gamma}) \geq 0$ by Lemma 5 part (d)(ii).

□

The proofs of Lemmas 2 - 4 apply Lemma 7 in order to show existence of caps or floors satisfying the relevant conditions out of (c2), (c2'), (c3), (c3'), (d2), or (d3). Lemma 7 tells us

that we need only check the relaxed conditions. That is, we need only confirm equalities or inequalities at single points rather than over an entire interval. This observation allows us to use arguments from continuity – i.e., the intermediate value theorem – to find the existence of such points.

Proof of Lemma 2. It holds that $G(\bar{\gamma}) = 0$, and by (12) there exists γ arbitrarily close to $\bar{\gamma}$ such that $G(\gamma) < 0$.⁴ Lemma 6 part (iii) therefore rules out $\pi^* = \pi_f(\bar{\gamma})$. Moreover, Lemma 6 part (iv) rules out $\pi^* > \pi_f(\bar{\gamma})$. So there are two possible cases:

$$(i) \quad \pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma})).$$

$$(ii) \quad \pi^* \leq \pi_f(\underline{\gamma}), \text{ i.e., } \hat{\gamma} \leq \underline{\gamma}.$$

In case (i), first observe that $H(\gamma) \leq 0$ for all γ ; this follows from $H(\underline{\gamma}) = 0$ (Lemma 5 part (a)(ii)), $H'(\underline{\gamma}) \leq 0$ (Lemma 5 part (b)(ii)), and H concave (Lemma 5 part (e)(ii)). In particular, $H(\hat{\gamma}) \leq 0$. So by Lemma 5 part (c), it must hold that $G(\hat{\gamma}) \geq 0$. Therefore, continuity of G implies that there exists $\gamma_H \in [\hat{\gamma}, \bar{\gamma})$ such that $G(\gamma_H) = 0$, i.e., such that (Rc2) holds. Now apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds $\underline{\gamma}, \gamma_H$ is optimal: (c1) is implied by (Gc1), (c2) by (Rc2) and Lemma 7 part (i), (c2') vacuously, (c3) vacuously, (c3') by assumption.

In case (ii), (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. Now apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal. \square

Proof of Lemma 3. It holds that $H(\underline{\gamma}) = 0$, and by (13) there exists γ arbitrarily close to $\underline{\gamma}$ such that $H(\gamma) > 0$.⁵ Lemma 6 part (ii) therefore rules out $\pi^* = \pi_f(\underline{\gamma})$. Moreover, Lemma 6 part (i) rules out $\pi^* < \pi_f(\underline{\gamma})$. So there are two possible cases:

$$(i) \quad \pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma})).$$

$$(ii) \quad \pi^* \geq \pi_f(\bar{\gamma}), \text{ i.e., } \hat{\gamma} \geq \underline{\gamma}.$$

In case (i), first observe that $G(\gamma) \geq 0$ for all γ ; this follows from $G(\bar{\gamma}) = 0$ (Lemma 5 part (a)(i)), $G'(\bar{\gamma}) \leq 0$ (Lemma 5 part (b)(i)), and G convex (Lemma 5 part (e)(i)). In particular, $G(\hat{\gamma}) \geq 0$. So by Lemma 5 part (c), it must hold that $H(\hat{\gamma}) \leq 0$. Therefore, continuity of H

⁴Note that the maintained assumption that F has full support on Γ allows for the possibility that $f(\bar{\gamma}) = 0$, and hence that $G'(\bar{\gamma}) = 0$ (Lemma 5 part (b)(i)). This is why we appealed directly to (12) to establish that there exists a point $\gamma < \bar{\gamma}$ with $G(\gamma) < 0$. Having established that fact, though, we can actually rule out $f(\bar{\gamma}) = 0$ under the assumptions of the Lemma. In particular, the combination of $G(\bar{\gamma}) = 0$; the existence of a point $\gamma < \bar{\gamma}$ with $G(\gamma) < 0$; and convexity of G , together imply that $G'(\bar{\gamma}) > 0$ and thus that $f(\bar{\gamma}) > 0$.

⁵Analogously to footnote 4 in the proof of Lemma 2, the assumptions of the Lemma imply that $H'(\gamma) > 0$ and thus that $f(\underline{\gamma}) = 0$.

implies that there exists $\gamma_L \in (\underline{\gamma}, \hat{\gamma}]$ such that $H(\gamma_L) = 0$, i.e., such that (Rc3) holds. Now apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds $\gamma_L, \bar{\gamma}$ is optimal: (c1) is implied by (Gc1), (c2) holds vacuously, (c2') by assumption, (c3) by (Rc3) and Lemma 7 part (ii), (c3') vacuously.

In case (ii), (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. Now apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal. \square

Proof of Lemma 4. It holds that $H(\underline{\gamma}) = G(\bar{\gamma}) = 0$. Moreover, there exists a point γ arbitrarily close to $\underline{\gamma}$ such that $H(\gamma) > 0$ (by (13)), and a point γ arbitrarily close to $\bar{\gamma}$ such that $G(\gamma) < 0$ (by (12)). Consider three cases:

(i) $\pi^* \leq \pi_f(\underline{\gamma})$, i.e., $\hat{\gamma} \leq \underline{\gamma}$.

(ii) $\pi^* \geq \pi_f(\bar{\gamma})$, i.e., $\hat{\gamma} \geq \bar{\gamma}$.

(iii) $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\bar{\gamma}))$, in which case $G(\underline{\gamma}) > 0$ and $H(\bar{\gamma}) < 0$ (by Lemma 5 part (d)).

In cases (i) and (ii), (Rd2) and (Rd3) hold vacuously, and therefore (d2) and (d3) hold by Lemma 7. For either of these cases, apply Proposition 1 of the current paper to get that the constant allocation π^* is optimal.

Finally, consider case (iii). By continuity, there must be some γ_L and γ_H in $(\underline{\gamma}, \bar{\gamma})$ such that $H(\gamma_L) = G(\gamma_H) = 0$. By concavity of H , it holds that $H(\gamma) > 0$ for all $\gamma \in (\underline{\gamma}, \gamma_L)$ and $H(\gamma) < 0$ for all $\gamma \in (\gamma_L, \bar{\gamma}]$. Likewise, by convexity of G , it holds that $G(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \gamma_H)$ and $G(\gamma) < 0$ for all $\gamma \in (\gamma_H, \bar{\gamma})$. In other words, for $\gamma < \min\{\gamma_L, \gamma_H\}$, it holds that $H(\gamma) \geq 0$ and $G(\gamma) > 0$, so $H(\gamma) + G(\gamma) > 0$. Hence, from Lemma 5 part (c), for any $\gamma < \min\{\gamma_L, \gamma_H\}$ it holds that $\pi^* > \pi_f(\gamma)$, i.e., that $\hat{\gamma} > \gamma$. Similarly, for $\gamma > \min\{\gamma_L, \gamma_H\}$ it holds that $H(\gamma) + G(\gamma) < 0$, and thus that $\pi^* < \pi_f(\gamma)$, i.e., that $\hat{\gamma} < \gamma$. Putting these observations together, $\hat{\gamma} \in [\min\{\gamma_L, \gamma_H\}, \max\{\gamma_L, \gamma_H\}]$.

Now consider two possibilities in case (iii). The first possibility is that $\gamma_L < \gamma_H$. Then we can apply Proposition 1 of Amador and Bagwell (2013) to get that the interval allocation with bounds γ_L, γ_H is optimal: (c1) is implied by (Gc1), (c2) holds by (Rc2) and Lemma 7 part (i), (c2') holds vacuously, (c3) holds by (Rc3) and Lemma 7 part (ii), and (c3') holds vacuously.

The second possibility is that $\gamma_L \geq \gamma_H$. Then $\pi^* \in [\pi_f(\gamma_H), \pi_f(\gamma_L)]$ or, in other words, $\hat{\gamma} \in [\gamma_H, \gamma_L]$. We have that (Rd2) holds because $\hat{\gamma} \geq \gamma_H$, and $G(\gamma) \leq 0$ for any $\gamma \geq \gamma_H$; and (Rd3) holds because $\hat{\gamma} \leq \gamma_L$, and $H(\gamma) \geq 0$ for any $\gamma \leq \gamma_L$. We can now apply Proposition 1 of the current paper to get that the constant allocation at π^* is optimal: (d2) holds by (Rc2) and Lemma 7 part (i), and (d3) holds by (Rd3) and Lemma 7 part (ii). \square