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RESEARCH ARTICLE

A note on interval delegation

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Abstract In this note we extend the Amador and Bagwell (Econometrica 81:1541–1599, 2013) conditions for confirming the optimality of a proposed interval delegation set to the possibility of degenerate intervals, in which the agent takes the same action at every state. We consider the cases of money burning as well as no money burning. These results allow us to provide new sufficient conditions on utility functions and state distributions to guarantee that some interval—degenerate or non-degenerate—will be optimal.

Keywords Delegation · Optimal contracting · Tariff caps

JEL Classification D82 · F10

1 Introduction

In a standard delegation problem, a state of the world determines both the principal's and agent's preferences over a one-dimensional action (see, e.g., Holmström 1977). The agent privately observes the state realization. The principal contracts with the agent by giving her a set of actions from which to choose. The problem may also be

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augmented by "money burning," an auxiliary action that hurts both players. A number of papers provide conditions to guarantee that an interval delegation set is optimal, and that—if money burning is available—no money is burned.[1] Such an interval may take the form of a cap on the actions of an upward-biased agent; a floor on the actions of a downward-biased agent; or a cap together with a floor, for an agent who takes actions that are sometimes too high and sometimes too low.

In many ways the most general sufficient conditions for the optimality of interval delegation, in terms of state distributions and utility functional form assumptions, come from Proposition 1 of Amador and Bagwell (2013). That paper considers problems both with and without money burning, and its utility functions embed those in earlier papers. However, Amador and Bagwell (2013) lack two kinds of results that may be important for readers. First, that paper does not consider degenerate intervals—single-point delegation sets. Second, Amador and Bagwell (2013) find the optimal interval, within the class of intervals, and then give conditions to verify that this proposed interval is optimal within the full class of incentive compatible allocations. But for some applications one might seek sufficient conditions for the optimality of some interval, without first identifying the exact boundaries of that interval.

This note addresses both issues. We first extend the results of Amador and Bagwell (2013) to present conditions for a proposed single point—a degenerate interval—to be an optimal delegation set. We then give sufficiency conditions to guarantee that some interval, degenerate or otherwise, must be optimal. It is enough to require that utility functions are in a certain family, one capturing many of those used in the literature, and that a global regularity condition jointly holds on the utilities and state distributions.

The two issues of degenerate intervals and global sufficiency conditions are inherently linked. The distributional and functional form assumptions which are sufficient to guarantee that some interval is optimal certainly can lead to a degenerate interval. For instance, an agent with a positive bias relative to the principal should be given a cap. But as the agent's bias grows stronger, the cap is eventually made to be always-binding. So one must address the possibility of degenerate intervals before moving to global sufficiency conditions.

Section 2 presents the result concerning the optimality of degenerate delegation intervals. Section 3 presents the sufficient conditions for the optimality of some interval delegation. Section 4 contains the proof of Proposition 1. An online appendix collects the rest of the proofs.

2 Optimality of degenerate delegation intervals

This paper considers the environment of Amador and Bagwell (2013), maintaining Assumption 1 of that paper throughout. Repeating the key definitions and assumptions, the principal's utility is given by $w(\gamma, \pi) - t$ and the agent's by $\gamma\pi + b(\pi) - t$. The

[1] Without money burning, see for instance Melumad and Shibano (1991), Martimort and Semenov (2006), and Alonso and Matouschek (2008). With money burning—captured, in the first two cases, via randomized actions—see Goltsman et al. (2009), Kováč and Mylovanov (2009), and Amador and Bagwell (2013). Ambrus and Egorov (2017) and Amador and Bagwell (2016) study related delegation problems in which the optimal policy may in fact involve some form of money burning incentive.

value $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$ represents the state of the world, drawn from a distribution F with continuous density function f and with full support over Γ.[2] The action is $\pi \in \Pi$, where Π is an interval of the real line with nonempty interior, with $\inf \Pi$ normalized to 0 and $\sup \Pi = \overline{\pi}$ (possibly infinite). The agent's preferred, or "flexible," action at state γ is indicated by $\pi_f(\gamma)$. Finally, $t \geq 0$ is the level of money burning.

Assumption 1 The following hold: (i) the function $w \colon \Gamma \times \Pi \to \mathbb{R}$ is continuous on $\Gamma \times \Pi$; (ii) for any $\gamma_0 \in \Gamma$, the function $w(\gamma_0, \cdot)$ is concave on Π and twice differentiable on $(0, \overline{\pi})$; (iii) the function $b \colon \Pi \to \mathbb{R}$ is strictly concave on Π, and twice differentiable on $(0, \overline{\pi})$; (iv) there exists a twice differentiable function $\pi_f : \Gamma \to (0, \overline{\pi})$ such that, for all $\gamma_0 \in \Gamma$, $\pi_f'(\gamma_0) > 0$ and $\pi_f(\gamma_0) \in \arg\max_{\pi \in \Pi}\{\gamma_0 \pi + b(\pi)\}$; and (v) the function $w_\pi : \Gamma \times (0, \overline{\pi}) \to \mathbb{R}$ is continuous on $\Gamma \times (0, \overline{\pi})$, where w_π denotes the derivative of w in its second argument.

The principal chooses an allocation rule $\boldsymbol{\pi} : \Gamma \to \Pi$ and a transfer rule $\boldsymbol{t} : \Gamma \to \mathbb{R}$ such that the agent, who privately observes the state, finds it incentive-compatible to report the state truthfully. We denote a pair $(\boldsymbol{\pi}, \boldsymbol{t})$ as an *allocation*. We consider two different problems:

- The *problem with money burning* is defined to be:

$$\max \int_\Gamma \big(w\big(\gamma, \boldsymbol{\pi}(\gamma)\big) - \boldsymbol{t}(\gamma)\big)\, \mathrm{d}F(\gamma) \quad \text{subject to:} \tag{P}$$
$$\gamma \in \arg\max_{\tilde{\gamma} \in \Gamma} \big\{\gamma \boldsymbol{\pi}(\tilde{\gamma}) + b\big(\boldsymbol{\pi}(\tilde{\gamma})\big) - \boldsymbol{t}(\tilde{\gamma})\big\}, \quad \text{for all } \gamma \in \Gamma$$
$$\boldsymbol{t}(\gamma) \geq 0, \quad \forall \gamma \in \Gamma$$

- The *problem without money burning* is Problem (P) with the additional constraint:

$$\boldsymbol{t}(\gamma) = 0, \quad \forall \gamma \in \Gamma. \tag{1}$$

Just as in Amador and Bagwell (2013), a key parameter is κ, which will be defined as

$$\kappa = \inf_{(\gamma,\pi) \in \Gamma \times \Pi} \left\{\frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)}\right\}, \quad \text{for the problem without money burning,} \tag{2}$$
$$\kappa = \min\left\{\inf_{(\gamma,\pi) \in \Gamma \times \Pi} \left\{\frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)}\right\}, 1\right\}, \quad \text{for the problem with money burning.} \tag{3}$$

A *constant allocation* is an allocation such that $\boldsymbol{\pi}(\gamma)$ is a constant function, and $\boldsymbol{t}(\gamma) = 0$, $\forall \gamma \in \Gamma$. The optimal allocation within the class of constant allocations is defined by a value π^* that solves

$$\pi^* \in \arg\max_\pi \mathbb{E}_{\tilde{\gamma} \sim F}[w(\tilde{\gamma}, \pi)].$$

We make the following assumption about π^*:

[2] Amador and Bagwell (2013) impose the stronger assumption that f is strictly positive over Γ. But Proposition 1 of that paper, the only result we will invoke, does not rely on strict positivity of f.

Assumption 2 π^* is in the interior of the action space Π.

The value π^* can, therefore, be characterized as a solution to the first-order condition:

$$\int_{\underline{\gamma}}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) \mathrm{d}\tilde{\gamma} = 0. \tag{4}$$

Define $\hat{\gamma} \in \mathbb{R}$ as follows:

$$\hat{\gamma} = -b'(\pi^*). \tag{5}$$

For any $\gamma \in \Gamma$, the first-order condition defining the agent's preferred action, $\pi_f(\gamma)$, is $\gamma = -b'(\pi_f(\gamma))$. So if $\hat{\gamma} \in \Gamma$, then $\hat{\gamma}$ is the state at which the agent's ideal point is π^*: $\hat{\gamma} = -b'(\pi_f(\hat{\gamma}))$, implying $\pi_f(\hat{\gamma}) = \pi^*$. However, it may be the case that $\hat{\gamma}$ lies outside the interval Γ, in which case π^* does not correspond to the agent's ideal point for any state. It holds that (i) $\hat{\gamma} \in \Gamma$ if and only if $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma})]$, and (ii) $\hat{\gamma}$ is strictly increasing in π^*.

Now define two conditions, (d2) and (d3), that will, for degenerate intervals, play the role of (c2) and (c3) from Amador and Bagwell (2013) for nondegenerate intervals:

(d2) If $\hat{\gamma} < \overline{\gamma}$,

$$(\gamma - \hat{\gamma})\kappa \geq \int_{\gamma}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi^*) \frac{f(\tilde{\gamma})}{1 - F(\gamma)} \mathrm{d}\tilde{\gamma}, \quad \forall \gamma \geq \hat{\gamma} \text{ in } \Gamma.$$

(d3) If $\hat{\gamma} > \underline{\gamma}$,

$$(\gamma - \hat{\gamma})\kappa \leq \int_{\underline{\gamma}}^{\gamma} w_\pi(\tilde{\gamma}, \pi^*) \frac{f(\tilde{\gamma})}{F(\gamma)} \mathrm{d}\tilde{\gamma}, \quad \forall \gamma \leq \hat{\gamma} \text{ in } \Gamma.$$

Amador and Bagwell (2013) had three other conditions that one might need to check: (c1), (c2$'$), and (c3$'$). Condition (c1) was relevant only for the interior of a delegation set, and conditions (c2$'$) and (c3$'$) applied to the non-binding edges of an interval; these conditions will not be relevant for degenerate intervals.

We now present one of our main results:

Proposition 1 *Optimality of degenerate intervals—sufficiency:*

(a) (No money burning) *If conditions* (d2) *and* (d3) *are satisfied with* κ *as defined in* (2), *then the constant allocation* π^* *solves the problem without money burning*, *that is*, *Problem* (P) *with the additional constraint* (1).
(b) (Money burning) *If conditions* (d2) *and* (d3) *are satisfied with* κ *as defined in* (3), *then the constant allocation* π^* *solves Problem* (P).

One can break the results into three cases. First, we may have $\hat{\gamma} \leq \underline{\gamma}$, which implies $\pi^* \leq \pi_f(\underline{\gamma})$ because b is concave. In this case we need only check (d2). That corresponds to a degenerate cap for an agent with a strong upward bias. In particular, the agent's ideal point satisfies $\pi_f(\gamma) \geq \pi^*$ for all $\gamma \in \Gamma$. Second, we may have

$\hat{\gamma} \geq \overline{\gamma}$, i.e., $\pi^* \geq \pi_f(\overline{\gamma})$, in which case we need only check (d3). That corresponds to a degenerate floor for an agent with a strong downward bias: $\pi_f(\gamma) \leq \pi^*$ for all $\gamma \in \Gamma$. Finally, $\hat{\gamma}$ may be interior in Γ, i.e., $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$, in which case we need to check both (d2) and (d3).

3 Sufficiency conditions for some interval to be optimal

While (c1) in Amador and Bagwell (2013) was not relevant for checking the optimality of degenerate intervals, some version of (c1) will indeed become relevant for confirming that some interval—degenerate or otherwise—is optimal. Let (Gc1) indicate global (c1), that is, a strengthening of condition (c1) to apply to all $\gamma \in \Gamma$ rather γ in some specified interval.

(Gc1) $\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma)) f(\gamma)$ is nondecreasing over $\gamma \in \Gamma$.

We now show that (Gc1) combined with a functional form assumption on w implies that some interval is optimal. Specifically, consider the functional form

$$w(\gamma, \pi) = A[b(\pi) + B(\gamma) + C(\gamma)\pi] \quad \text{with } A > 0. \tag{6}$$

When there is no money burning, the value of A can be set to 1 without loss of generality. As shown by Amador and Bagwell (2013), this preference specification encompasses several prominent specifications found in the literature.[3]

With w of the form (6), the value of κ given by (2), without money burning, is A. The value of κ given by (3), with money burning, is $\min\{A, 1\}$. When we consider money burning below, we make the additional assumption that $A \leq 1$, in which case $\kappa = A$. Moreover, Assumption 1(iii) states that b is strictly concave, and hence w is strictly concave in π (Assumption 1(ii) had only imposed weak concavity). Therefore, $\pi^* \in \arg\max_\pi \mathbb{E}_{\tilde{\gamma}\sim F}[w(\tilde{\gamma}, \pi)]$ will be uniquely defined.

Recall that $w_\pi(\gamma, \pi_f(\gamma))$ gives the agent's bias at state γ: a negative value indicates that the agent is biased upwards relative to the principal, and a positive value that the agent is biased downwards. In what follows below, we present four lemmas that cover the two-by-two exhaustive cases of an upwards or downwards bias at the lowest state; and an upwards or downwards bias at the highest state.

For the following Lemmas, given $\gamma_L < \gamma_H$ in Γ, an *interval allocation with bounds* γ_L, γ_H refers to a nondegenerate interval delegation set in which the agent can select any action in the range $[\pi_f(\gamma_L), \pi_f(\gamma_H)]$, and there is no money burning:

Lemma 1 (Unconstrained interval) *Let w be of the form* (6). *Suppose that* (Gc1) *holds with $\kappa = A$. Suppose that $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$ and $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) \geq 0$. Then the*

[3] For example, the preferences assumed by Alonso and Matouschek (2008)—which does not allow for money burning—are a special case. Restating that conclusion, the primary model of Alonso and Matouschek (2008) proposes an agent utility function of $v_A(\pi - \pi_f(\gamma), \gamma)$, for v_A single-peaked and symmetric about 0 in its first term; a principal utility of $-r(\gamma)(\pi - \pi_P(\gamma))^2$, for r everywhere positive; and no money burning. Without loss, they can transform the problem so that $\pi_f(\gamma) = \gamma$ and $r(\gamma) = 1$. Indeed, the agent's behavior given any delegation set is as if her utility function were $\gamma\pi + b(\pi)$ for $b(\pi) = -\pi^2/2$. The principal's utility can then be written as $w(\gamma, \pi)$ as in (6) with $A = 1$, $C(\gamma) = \pi_P(\gamma)$, and $B(\gamma) = -(\pi_P(\gamma))^2/2$.

solution to Problem (P) *with additional constraint* (1); *or*, *under the further assumption that* $A \leq 1$, *the solution to Problem* (P); *is given by the interval allocation with bounds* $\underline{\gamma}, \overline{\gamma}$. *Moreover*, *it holds that* $\pi^* \in [\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma})]$.

Lemma 2 (Cap) *Let* w *be of the form* (6). *Suppose that* (Gc1) *holds with* $\kappa = A$. *Suppose that* $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$ *and* $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) < 0$. *Then the solution to Problem* (P) *with additional constraint* (1); *or*, *under the further assumption that* $A \leq 1$, *the solution to Problem* (P); *is as follows*:

(i) *If* $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$: *The interval allocation with bounds* $\underline{\gamma}, \gamma_H$ *is optimal*, *for some* $\gamma_H \in [\hat{\gamma}, \overline{\gamma})$.
(ii) *If* $\pi^* \leq \pi_f(\underline{\gamma})$: *The constant allocation* π^* *is optimal.*

It cannot hold that $\pi^* \geq \pi_f(\overline{\gamma})$.

Lemma 3 (Floor) *Let* w *be of the form* (6). *Suppose that* (Gc1) *holds with* $\kappa = A$. *Suppose that* $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$ *and* $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) \geq 0$. *Then the solution to Problem* (P) *with additional constraint* (1); *or*, *under the further assumption that* $A \leq 1$, *the solution to Problem* (P); *is as follows*:

(i) *If* $\pi^* \in (\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma}))$: *The interval allocation with bounds* $\gamma_L, \overline{\gamma}$ *is optimal*, *for some* $\gamma_L \in (\underline{\gamma}, \hat{\gamma}]$.
(ii) *If* $\pi^* \geq \pi_f(\overline{\gamma})$: *The constant allocation* π^* *is optimal.*

It cannot hold that $\pi^* \leq \pi_f(\underline{\gamma})$.

Lemma 4 (Cap and Floor) *Let* w *be of the form* (6). *Suppose that* (Gc1) *holds with* $\kappa = A$. *Suppose that* $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) > 0$ *and* $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma})) < 0$. *Then the solution to Problem* (P) *with additional constraint* (1); *or*, *under the further assumption that* $A \leq 1$, *the solution to Problem* (P); *is given by one of the following cases*:

(i) *The interval allocation with bounds* γ_L, γ_H *is optimal*, *for some* γ_L, γ_H *satisfying* $\underline{\gamma} < \gamma_L < \gamma_H < \overline{\gamma}$. *In this case it must hold that* $\pi^* \in [\pi_f(\gamma_L), \pi_f(\gamma_H)]$.
(ii) *The constant allocation* π^* *is optimal.*

The proofs of these lemmas build off of an argument in Amador and Bagwell (2016, Lemma 1). Under the functional form assumption (6), rather than checking (c2), (c3), (d2) and (d3) over an interval, condition (Gc1) allows us to check the conditions just at the boundary point. As we explain in the proofs, a related argument is developed by Alonso and Matouschek (2008). For the payoffs that they consider, condition (Gc1) implies the convexity or concavity of their "forward bias" and "backward bias" expressions (which play an important role in their argument).

Lemma 1 gives conditions for the optimality of an unconstrained delegation interval, with no binding cap or floor. This occurs when the agent is "more moderate" than the principal, being biased towards higher actions at the lowest state and lower actions at the highest state. Lemma 2 gives conditions for a cap, possibly binding for only some states (part (i)) and possibly degenerate and binding for all states (part (ii)). This occurs when the agent is biased upwards at the lowest and highest states. When the cap is not always binding (part (iii)), the cap is above the principal's ex ante optimal action π^*.

Lemma 3 gives similar conditions for a floor, when the agent is biased downwards at the lowest and highest states.

Lemma 4 gives conditions for interval delegation in the remaining case, in which the agent is biased downwards at the lowest state and upwards at the highest state. This can be thought of as an "extreme-biased" agent, although it is also consistent with simple misalignment of preferences: the principal's ideal point may be falling over some range of states as the agent's ideal point rises. Part (i) describes the case of a floor combined with a cap, where the floor is strictly below the cap. Part (ii) actually combines multiple cases. It may be that $\pi^* \leq \pi_f(\underline{\gamma})$, in which case the contract is an always-binding cap. It may be that $\pi^* \geq \pi_f(\overline{\gamma})$, in which case the contract is an always-binding floor. Or it may be that π^* is in between $\pi_f(\underline{\gamma})$ and $\pi_f(\overline{\gamma})$, in which case the contract can be thought of as a cap and a floor set at the same point.

Putting together these four lemmas:

Proposition 2 *Let w be of the form* (6). *Suppose that* (Gc1) *holds with $\kappa = A$.*

(a) (*No money burning*) *Then the solution to Problem* (P) *with additional constraint* (1) *is given by some interval delegation set, possibly degenerate.*
(b) (*Money burning*) *Suppose also that $A \leq 1$. Then the solution to Problem* (P) *is given by some interval delegation set, possibly degenerate.*

4 Proof of Proposition 1

We follow closely the proof of Proposition 1 in Amador and Bagwell (2013). The main difference is the proposed Lagrange multipliers, as in this case we are allowing for the possibility that the optimal allocation involves no flexibility. Once these multipliers have been found, the steps of the proof are identical. See Amador and Bagwell (2013) for a more detailed discussion of these steps. We will prove each of the parts of this Proposition separately.

Proof of part (a) of Proposition 1: without money burning

We first write the incentive constraints in their usual monotonicity restriction plus an integral form. Just as in Amador and Bagwell (2013), we write the integral form as two inequalities:

$$\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})\mathrm{d}\tilde{\gamma} + \underline{U} - \gamma\boldsymbol{\pi}(\gamma) - b(\boldsymbol{\pi}(\gamma)) \leq 0, \quad \text{for all } \gamma \in \Gamma, \tag{7}$$

$$-\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})\mathrm{d}\tilde{\gamma} - \underline{U} + \gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) \leq 0, \quad \text{for all } \gamma \in \Gamma, \tag{8}$$

where $\underline{U} \equiv \underline{\gamma}\boldsymbol{\pi}(\underline{\gamma}) + b(\boldsymbol{\pi}(\underline{\gamma}))$.

The problem is then to choose a function $\boldsymbol{\pi} \in \Phi$ so as to maximize

$$\max_{\boldsymbol{\pi}:\,\Gamma\to\Pi} \int w(\gamma, \boldsymbol{\pi}(\gamma))\mathrm{d}F(\gamma)$$

subject to (7) and (8) and where the choice set incorporates the monotonicity restriction: $\Phi \equiv \{\boldsymbol{\pi} | \boldsymbol{\pi} : \Gamma \to \Pi$ and $\boldsymbol{\pi}$ non-decreasing$\}$.

We then assign cumulative Lagrange multiplier functions Λ_1 and Λ_2 to constraints (7) and (8), respectively, and write the Lagrangian for the problem and after integrating by parts we obtain

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi}|\Lambda) = & \int_\Gamma [w(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) - (\Lambda(\overline{\gamma}) - \Lambda(\gamma))\boldsymbol{\pi}(\gamma)]\mathrm{d}\gamma \\ & + \int_\Gamma (\gamma \boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))\mathrm{d}\Lambda(\gamma) - \underline{U}(\Lambda(\overline{\gamma}) - \Lambda(\underline{\gamma})),\end{aligned}$$

where $\Lambda(\gamma) \equiv \Lambda_1(\gamma) - \Lambda_2(\gamma)$.

A proposed multiplier Let us propose some non-decreasing multipliers Λ_1 and Λ_2 so that their difference, Λ, satisfies:

$$\Lambda(\gamma) = \begin{cases} 1 & \gamma = \overline{\gamma} \\ 1 + \kappa(1 - F(\gamma)) & \gamma \in (\gamma^*, \overline{\gamma}) \\ 1 + \kappa(1 - F(\gamma^*)) & \gamma = \gamma^* \\ 1 - \kappa F(\gamma) & \gamma \in (\underline{\gamma}, \gamma^*) \\ 1 & \gamma = \underline{\gamma} \end{cases} \quad \text{and} \quad \gamma^* \in (\underline{\gamma}, \overline{\gamma}),$$

where κ is given by (2) and $\gamma^* = \min\left\{\max\{\hat{\gamma}, \underline{\gamma}\right\}, \overline{\gamma}\} \in \Gamma$ where $\hat{\gamma}$ is given by (5).[4] Note that Λ is well defined even when $\hat{\gamma}$ lies outside $[\underline{\gamma}, \overline{\gamma}]$.

Below we show that $\kappa F(\gamma) + \Lambda(\gamma) \equiv R(\gamma)$ is non-decreasing; hence, it follows that $\Lambda(\gamma)$ can indeed be written as the difference of two non-decreasing functions, $R(\gamma) - \kappa F(\gamma)$.

Concavity of the Lagrangian The next step is to check that the Lagrangian is concave when evaluated at the proposed multipliers. Towards this goal, we first note that the jump at γ^* in Λ equals κ, and thus is non-negative. Using that $\Lambda(\overline{\gamma}) = \Lambda(\underline{\gamma}) = 1$, we can write the Lagrangian as follows:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi}|\Lambda) = & \int_\Gamma [w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma)\mathrm{d}\gamma - \int_\Gamma (1 - \Lambda(\gamma))\boldsymbol{\pi}(\gamma)\mathrm{d}\gamma \\ & + \int_\Gamma (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))\mathrm{d}(\kappa F(\gamma) + \Lambda(\gamma)). \end{aligned} \tag{9}$$

The concavity of $w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa b(\boldsymbol{\pi}(\gamma))$ in $\boldsymbol{\pi}(\gamma)$ follows from the definition of κ. The fact that its jump at γ^* is non-negative implies that $\kappa F(\gamma) + \Lambda(\gamma)$ is non-decreasing.

Maximizing the Lagrangian We now proceed to show that the proposed allocation $\boldsymbol{\pi}^*$ maximizes the Lagrangian.

[4] Here we use the convention that the open interval (x, x) corresponds to the empty set, $\emptyset$.

As in Amador and Bagwell (2013), we extend b and w to the entire positive ray of the real line. Denoting $\hat{\Phi} = \{\boldsymbol{\pi} | \boldsymbol{\pi} : \Gamma \to \mathbb{R}_+ \text{ and } \boldsymbol{\pi} \text{ non-decreasing}\}$, we can say that if

$$\begin{aligned}
&\partial \mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{\pi}^* | \Lambda) = 0, \\
&\partial \mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{x} | \Lambda) \leq 0; \quad \text{for all } \boldsymbol{x} \in \hat{\Phi},
\end{aligned}$$

then $\boldsymbol{\pi}^*$ maximizes the Lagrangian $\mathcal{L}$.

For our problem, taking the Gateaux differential in direction $\boldsymbol{x} \in \hat{\Phi}$. we get that:[5]

$$\begin{aligned}
\partial \mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{x} | \Lambda) = &\int_{\Gamma} \left[w_\pi(\gamma, \pi^*) f(\gamma) - (1 - \Lambda(\gamma)) \right] \boldsymbol{x}(\gamma) \mathrm{d}\gamma \\
&+ \int_{\underline{\gamma}}^{\overline{\gamma}} \left(\gamma - \hat{\gamma}\right) \boldsymbol{x}(\gamma) \mathrm{d}\Lambda(\gamma)
\end{aligned} \tag{10}$$

which can be rewritten as

$$\begin{aligned}
\partial \mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{x} | \Lambda) = &\int_{\underline{\gamma}}^{\gamma^*} \left[w_\pi(\gamma, \pi^*) f(\gamma) - \kappa F(\gamma) - \kappa(\gamma - \hat{\gamma}) f(\gamma) \right] \boldsymbol{x}(\gamma) \mathrm{d}\gamma \\
&+ \int_{\gamma^*}^{\overline{\gamma}} \left[w_\pi(\gamma, \pi^*) f(\gamma) + \kappa(1 - F(\gamma)) - \kappa(\gamma - \hat{\gamma}) f(\gamma) \right] \boldsymbol{x}(\gamma) \mathrm{d}\gamma \\
&+ \kappa(\gamma^* - \hat{\gamma}) \boldsymbol{x}(\gamma^*)
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
\partial \mathcal{L}(\boldsymbol{\pi}^*; \boldsymbol{x} | \Lambda) = \boldsymbol{x}(\gamma^*) \Bigg\{ &\int_{\underline{\gamma}}^{\gamma^*} \left[w_\pi(\gamma, \pi^*) f(\gamma) - \kappa F(\gamma) - \kappa(\gamma - \hat{\gamma}) f(\gamma) \right] \mathrm{d}\gamma \\
&+ \int_{\gamma^*}^{\overline{\gamma}} \left[w_\pi(\gamma, \pi^*) f(\gamma) + \kappa(1 - F(\gamma)) - \kappa(\gamma - \hat{\gamma}) f(\gamma) \right] \mathrm{d}\gamma + \kappa(\gamma^* - \hat{\gamma}) \Bigg\} \\
&- \int_{\underline{\gamma}}^{\gamma^*} \left[\int_{\underline{\gamma}}^{\gamma} \left[w_\pi(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) - \kappa F(\tilde{\gamma}) - \kappa(\tilde{\gamma} - \hat{\gamma}) f(\tilde{\gamma}) \right] \mathrm{d}\tilde{\gamma} \right] \mathrm{d}\boldsymbol{x}(\gamma) \\
&+ \int_{\gamma^*}^{\overline{\gamma}} \left[\int_{\gamma}^{\overline{\gamma}} \left[w_\pi(\tilde{\gamma}, \pi^*) f(\tilde{\gamma}) + \kappa(1 - F(\tilde{\gamma})) - \kappa(\tilde{\gamma} - \hat{\gamma}) f(\tilde{\gamma}) \right] \mathrm{d}\tilde{\gamma} \right] \mathrm{d}\boldsymbol{x}(\gamma).
\end{aligned} \tag{11}$$

We require that this differential be non-positive for all non-decreasing $\boldsymbol{x}$ and zero when evaluated at $\boldsymbol{x} = \boldsymbol{\pi}^*$. Note that if $\boldsymbol{x} = \boldsymbol{\pi}^*$, then $\mathrm{d}\boldsymbol{x}(\gamma) = 0$. So we have that

[5] Existence of the Gateaux differentials and the ability to integrate by parts follows from identical arguments to those in footnotes 52 and 53 of Amador and Bagwell (2013).

$$
\begin{aligned}
\partial\mathcal{L}(\boldsymbol{\pi}^*;\boldsymbol{\pi}^*|\Lambda) = \pi^*\Bigg\{&\int_{\underline{\gamma}}^{\gamma^*}\big[w_\pi(\gamma,\pi^*)f(\gamma)-\kappa F(\gamma)-\kappa(\gamma-\hat{\gamma})f(\gamma)\big]\mathrm{d}\gamma\\
&+\int_{\gamma^*}^{\overline{\gamma}}\big[w_\pi(\gamma,\pi^*)f(\gamma)+\kappa(1-F(\gamma))-\kappa(\gamma-\hat{\gamma})f(\gamma)\big]\mathrm{d}\gamma\\
&+\kappa(\gamma^*-\hat{\gamma})\Bigg\}\\
&=\kappa\pi^*\left[\int_{\underline{\gamma}}^{\overline{\gamma}}\big[-F(\gamma)-\gamma f(\gamma)\big]\mathrm{d}\gamma+\overline{\gamma}\right]=0,
\end{aligned}
$$

where the last equality follows from (4) and the identity $\int_a^b \gamma f(\gamma)\mathrm{d}\gamma = bF(b) - aF(a) - \int_a^b F(\gamma)\mathrm{d}\gamma$.

To guarantee that $\partial\mathcal{L}(\boldsymbol{\pi}^*;\boldsymbol{x}|\Lambda)\le 0$ for all $\boldsymbol{x}\in\hat{\Phi}$, we need that

$$
\begin{aligned}
&\int_{\underline{\gamma}}^{\gamma}\big[w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})-\kappa F(\tilde{\gamma})-\kappa(\tilde{\gamma}-\hat{\gamma})f(\tilde{\gamma})\big]\mathrm{d}\tilde{\gamma}\ge 0 \quad \text{for all } \gamma\in[\underline{\gamma},\gamma^*)\\
&\int_{\gamma}^{\overline{\gamma}}\big[w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})+\kappa(1-F(\tilde{\gamma}))-\kappa(\tilde{\gamma}-\hat{\gamma})f(\tilde{\gamma})\big]\le 0 \quad \text{for all } \gamma\in(\gamma^*,\overline{\gamma}],
\end{aligned}
$$

where this follows by noticing $\partial\mathcal{L}(\boldsymbol{\pi}^*;\boldsymbol{\pi}^*|\Lambda) = 0$ implies that the term in curly brackets in Eq. (11) equals zero.

These conditions in turn are implied by (d2) and (d3). To see, note that

$$
\begin{aligned}
&\int_{\underline{\gamma}}^{\gamma}[w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})-\kappa F(\tilde{\gamma})-\kappa(\tilde{\gamma}-\hat{\gamma})f(\tilde{\gamma})]\mathrm{d}\tilde{\gamma}\\
&=\int_{\underline{\gamma}}^{\gamma}w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})\mathrm{d}\tilde{\gamma}-\kappa(\gamma-\hat{\gamma})F(\gamma)
\end{aligned}
$$

And thus (d3), together with the definition of γ^*, implies that

$$
\int_{\underline{\gamma}}^{\gamma}\big[w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})-\kappa F(\tilde{\gamma})-\kappa(\tilde{\gamma}-\hat{\gamma})f(\tilde{\gamma})\big]\mathrm{d}\tilde{\gamma}\ge 0 \quad \text{for all } \gamma\in[\underline{\gamma},\gamma^*)
$$

A similar argument shows that (d2) implies

$$
\int_{\gamma}^{\overline{\gamma}}\big[w_\pi(\tilde{\gamma},\pi^*)f(\tilde{\gamma})+\kappa(1-F(\tilde{\gamma}))-\kappa(\tilde{\gamma}-\hat{\gamma})f(\tilde{\gamma})\big]\le 0 \quad \text{for all } \gamma\in(\gamma^*,\overline{\gamma}]
$$

Hence, using concavity of Lagrangian plus Lemma A.2 in Amador et al. (2006), we have shown that the proposed allocation $\boldsymbol{\pi}^*$ maximizes the Lagrangian given the multipliers.

Applying Luenberger's sufficiency theorem Just as in Amador and Bagwell (2013), we then apply Theorem 1 in their appendix to show that π^* is an optimal solution of the original problem.

Proof of part (b) of Proposition 1: with money burning

The proof of part (b) follows the same steps as in Amador and Bagwell (2013), but this time with the multiplier given by

$$\tilde{\Lambda}(\gamma) = \begin{cases} 1 & \gamma = \overline{\gamma} \\ (1-\kappa)F(\gamma)+\kappa & \gamma \in (\gamma^*, \overline{\gamma}) \\ (1-\kappa)F(\gamma^*)+\kappa & \gamma = \gamma^* \\ (1-\kappa)F(\gamma) & \gamma \in (\underline{\gamma}, \gamma^*) \\ 0 & \gamma = \underline{\gamma} \end{cases} \quad \text{and} \quad \gamma^* \in (\underline{\gamma}, \overline{\gamma}),$$

where κ is given by definition (3), and $\gamma^* = \min\{\max\{\hat{\gamma}, \underline{\gamma}\}, \overline{\gamma}\} \in \Gamma$ where $\hat{\gamma}$ is given by (5).

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