Pareto Improving Fiscal and Monetary Policies: Samuelson in the New Keynesian Model*

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Abstract

This paper explores the positive and normative consequences of government bond issuances in a New Keynesian model with heterogeneous agents, focusing on how the stock of government bonds affects the cross-sectional allocation of resources in the spirit of Samuelson (1958). We characterize the Pareto optimal levels of government bonds and the associated monetary policy adjustments that should accompany Pareto-improving bond issuances. The paper introduces a simple phase diagram to analyze the global equilibrium dynamics of inflation, interest rates, and labor earnings in response to changes in the stock of government debt. The framework also provides a tractable tool to explore the use of fiscal policy to escape the Effective Lower Bound (ELB) on nominal interest rates and the resolution of the "forward guidance puzzle." A common theme throughout is that following the monetary policy guidance from the standard Ricardian framework leads to excess fluctuations in income and inflation.

1 Introduction

It is well known, going back to at least Samuelson (1958), that the introduction of outside assets can (Pareto) improve the allocation of consumption in models with heterogeneity. The role of

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government bonds as a safe store of value, in particular, has played an important role in welfare analyses of both over-lapping generations (OLG) and Bewley-Huggett-Aiyagari incomplete market economies. We revisit this idea in a tractable OLG version of the standard New Keynesian model. In particular, we analyze the positive and normative consequences of government bond issuances, and how these depend on the monetary policy rule, both at and away from the Effective Lower Bound (ELB) on nominal interest rates. The main tool of analysis is a phase diagram that allows easy analysis of the economy's response to a variety of shocks, without recourse to local approximations. We show that the standard monetary rule that is derived in the Ricardian setting generates unnecessary fluctuations in labor earnings, markups, and inflation in response to a potentially welfare-improving bond issuance. The focus on Pareto improvements complements the explicitly redistributional policies that have been the focus of the recent policy analyses in heterogeneous agent New Keynesian (HANK) models.[1]

A few distinguishing characteristics of the approach should be flagged at the outset. The analysis abstracts from the direct link between monetary policy and the government budget constraint studied in the classic paper of Sargent and Wallace (1981) and the more recent work on the fiscal theory of the price level (e.g., Cochrane, 2023). To make the distinction from the latter literature crystal clear, we model the government as issuing real bonds, although this is done for expositional reasons rather than as a necessary component of the analysis. To contrast with work of Sargent and Wallace (1981), we explore a cashless economy with zero seigniorage revenue.

We build on two standard platforms. To generate a link between government debt and the real economy, we break Ricardian Equivalence using the perpetual youth framework of Blanchard (1985) and Yaari (1965), augmented to include endogenous labor supply but without physical capital. In this framework, we embed a textbook New Keynesian (NK) model of nominal rigidities, as in Rotemberg (1982), Galí (2015), and Woodford (2004). This is a powerful combination that has been used several times in the literature, which we review below. It also generates heterogeneity in a tractable manner, providing clean insights that are applicable to richer heterogeneous agent New Keynesian models now popular in the quantitative literature.

One useful assumption we make is that the individuals supplying labor (workers) save (in aggregate) in the government bond, while firm owners (entrepreneurs) save in equity. This dichotomy echoes the reality that many households whose primary income consists of labor earnings do not own shares of firms. As a modeling choice, the key implication is that price setters and holders of government bonds value inter-temporal tradeoffs differently, as there may be a wedge between the equilibrium return on government bonds and the return on equity. While

[1]Recent examples of optimal policy in HANK models using a utilitarian criteria include Bhandari et al. (2020), Nuno and Thonmas (2021), Dávilla and Schaab (2022), Acharya, Challe, and Dogra (2020), Bilbiie and Ragot (2021), and McKay and C. Wolf (2022) and Yang (2022). LeGrand, Martin-Baillon, and Ragot (2021) studies optimal policy using a social welfare function with empirically derived weights.

stark, this segmentation proves extremely convenient when analyzing dynamics.

To provide a sense of the equilibrium, consider first the government bond market. Workers' willingness to hold bonds is an increasing function of the real return, an increasing function of their real income (labor earnings), and a decreasing function of anticipated income growth. An increase in government debt requires some combination of a shift in these three equilibrium outcomes to clear the bond market.

In the product market, prices are set in a monopolistically competitive fashion subject to quadratic adjustment costs, generating a standard NK Phillips curve that relates anticipated inflation, current inflation, and the real wage (inverse markup). Finally, monetary policy is set through a nominal interest rate that satisfies a Taylor rule, subject to an effective/zero lower bound (ELB). Specifically, the nominal interest rate is a linear function of inflation, with a slope coefficient strictly greater than one (i.e., the Taylor Principle holds).

Given these equilibrium restrictions, for a given level of government debt, we characterize the economy as a system of two non-linear ordinary differential equations (ODEs). We do so in terms of inflation and the real wage (or inverse markup). The system of ODEs is amenable to analysis using a simple phase diagram. We show that there are two steady states.[2] One is unstable and represents the monetary authorities target inflation rate and the flexible price real wage. The other is a stable Effective Lower Bound (ELB) equilibrium at which the nominal interest rate is zero, which corresponds to the equilibrium studied by Benhabib, Schmitt-Grohé, and Uribe (2001).

We use the phase diagram to explore the consequences of fiscal and monetary policies. In particular, starting from the unstable steady state, consider a surprise announcement at $t = 0$ of one-time debt-financed tax cut (or transfer) that will take place in $t' > 0$ periods. This is equivalent to a "helicopter" distribution of government bonds to tax payers at t'. We show that if t' is not too distant, inflation, both the nominal and real interest rates, and real labor earnings will jump up on announcement. After the initial impact, inflation, interest rates, and labor income will all continue to increase reaching the new steady state at precisely $t = t'$.[3] At this new steady state, inflation, interest rates, and labor earnings are higher.

To understand these dynamics, the higher real interest rate at t' is necessary to clear the bond market at a larger stock of government debt. That is, in our non-Ricardian environment, there is a positive steady-state relationship between the amount of government bonds held by savers and the equilibrium real interest rate. If the monetary authority follows a fixed Taylor rule that increases the nominal interest rate more than one-for-one with inflation, then higher inflation corresponds to a larger real interest rate, and vice versa. Hence, the new steady state

[2]There area potentially other steady states, but they are uninteresting and we defer discussion of it until the body of the paper.

[3]It is not obvious whether or why the economy must be at the new steady state at t', but we postpone a detailed discussion of this to the body of the paper.

has higher inflation, and, via the Phillips curve, a higher level of labor income. Along the path to the new steady state, inflation begins to accelerate immediately upon announcement to avoid an anticipated discontinuity in inflation at t'. This trajectory coincides with accelerating labor earnings, as well. For a short horizon, the response of the economy to an anticipated fiscal deficit is therefore an immediate increase in inflation and labor income and growth in both to the new steady state.

If t' is far enough in the future, the initial impact may actually be a *decrease* in labor income. That is, the announcement of a future tax cut may lower labor income on impact, although consumption demand remains strong due to the anticipation of future income growth. Inflationary pressure induces the monetary authority to raise nominal (and real) interest rates, as before, but to clear the bond market at announcement a reduction in worker income is also necessary. Thereafter, labor income, inflation, and interest rates accelerate to reach the new steady state at t'.

This assumes that the monetary policy *rule* remains invariant to fiscal policy, as in the standard prescription derived using Ricardian models.[4] We show that if the Taylor rule intercept is anticipated to shift at the same time the fiscal authority issues new debt, then the dynamics are muted. In the case that the monetary authority announces that it will shift the nominal rate one-for-one with the zero-inflation real rate at t', there are no dynamics until t', at which point government bonds, the nominal interest rate, and the real rate increase simultaneously with no change in inflation or real wages. The analysis highlights the importance of adapting policy for non-Ricardian environments. If the central bank follows the policy advice obtained from the Ricardian benchmark (in which the steady-state real interest rate is independent of fiscal policy) it will induce unnecessary fluctuations in both income and inflation in response to fiscal deficits. In particular, the central bank will appear to be conscientiously fighting the inflation apparently caused by the lax fiscal authority, when in reality the rigidity of the monetary policy rule is equally to blame for the adverse consequences.

With the positive analysis in hand, we turn to whether expanding the stock of government bonds can improve welfare. This has been the focus of much work using real models in the tradition of Aiyagari (1994). We show that there is a target real interest rate that maximizes the stationary welfare of agents, which extends the classic insight of Samuelson (1958) and Balasko and Shell (1980). However, issuing bonds to achieve this level may be inflationary in the NK framework. We show that the monetary authority must work in a complementary fashion to fiscal policy in order to achieve the Pareto improvement. In doing so, we relate the Pareto improvement in the OLG-New Keynesian model to the Robust Pareto Improvements (RPI) we

[4]More precisely, in Ricardian models the long-run real interest rate is pinned down independently of debt, and hence bond issuances do not alter the long-run target nominal interest rate. This is separate from secular shifts in preferences (discount factors), openness, or demographic changes that may play a role in long-run trends in the real interest rate.

previously explored in Aguiar, Amador, and Arellano (2022) using the Aiyagari model.

While our focus is primarily on how and when government bonds improve allocations in economies with heterogeneity and nominal rigidities, the framework also allows a tractable analysis of two other questions that have been core topics in monetary economics. The first concerns a transitory decline in the discount rate of savers, which is a popular device for generating an exogenous decline in "aggregate demand." If the ELB does not bind, we show that the economy experiences a decline in inflation and labor earnings on impact, and then a recovery that features rising inflation and potentially non-monotonic dynamics in earnings. Again, the intuition can be obtained from bond market equilibrium; the increased desire to save must be accommodated through lower real interest rates, lower real income, or anticipated income growth. From the Taylor rule, lower real interest rates are generated via lower inflation. The Phillips curve then requires a decline in real wages.

It may be the case that the increase in patience is so severe that the ELB binds. In this case, we show that fiscal policy provides a better alternative to the traditional "forward guidance" of Krugman (1998), Eggertsson and Woodford (2003), and Werning (2007). In particular, the fiscal authority can reflate the economy by issuing bonds and rebating the proceeds back to tax payers. The effectiveness of this policy, of course, leverages the non-Ricardian environment.

Another extension relates to the forward guidance puzzle of Del Negro, Giannoni, and Patterson (2023) and McKay, Nakamura, and Steinsson (2016). This puzzle stems from the fact that in the standard representative agent New Keynesian model, an anticipated decline in interest rates far in the future has a large impact on current demand. Our phase diagram shows clearly why, in a non-Ricardian environment there is no such puzzle. The analysis provides additional insight into the resolution of the puzzle proposed by Del Negro, Giannoni, and Patterson (2023), without relying on linear approximations.

Our interest in the interplay of fiscal policy, interest rates, and inflation is motivated in part by the large swings in public debt as a percentage of national income observed in many advanced economies. In Figure 1, we plot public debt as a fraction of GDP for the United States between 1960 and 2022. As is well known, there were sharp increases during the Reagan and George W. Bush administrations as well as during the Great Recession and the COVID-19 pandemic. Conversely, debt declined during the Clinton administration. Figure 1 also plots inflation (dotted line) and a measure of the 5-year real interest rate (dashed line) that controls for demographic trends.[5] There is no clear link between inflation, real rates, and debt.

A little more nuance can be found by looking at two episodes. Panel A of Figure 2 is a scatter

[5] Inflation is the four quarter change in the core PCE index. The 5-year real interest rate is defined as the difference between the 5-year Treasury yield and the average 5-year inflation expectations, residualized with life expectancy and fertility. These inflation expectations are based on the average quarterly inflation forecasts over the next 5 years using current inflation.

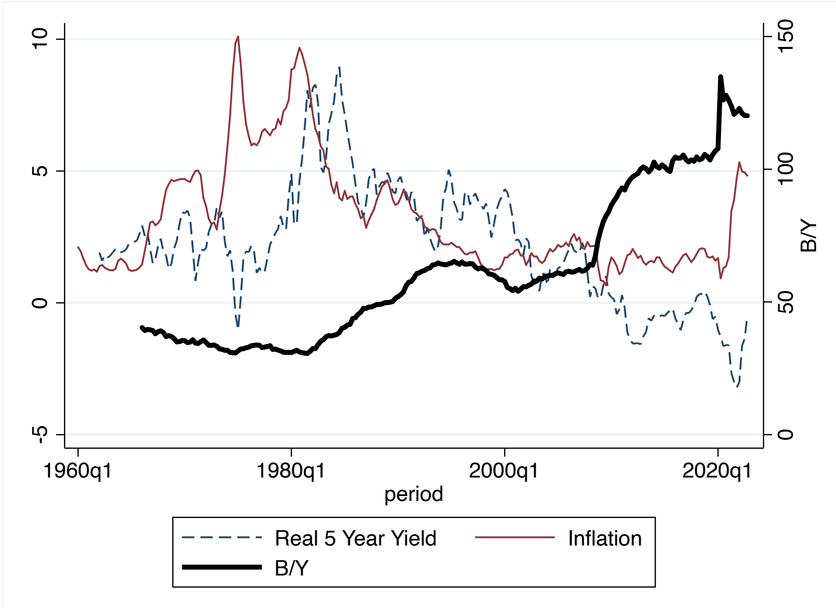


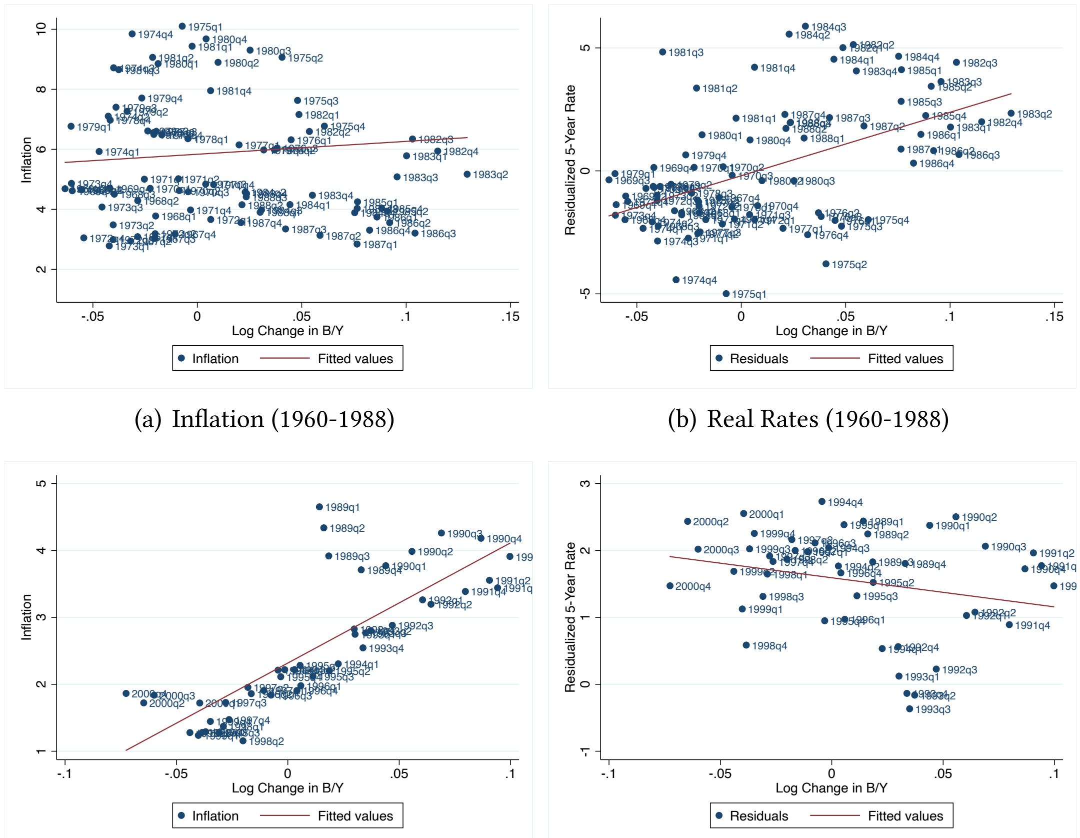
Figure 1: Public Debt, Inflation, and Real Rates in the U.S.

plot of the annual log change in debt-to-GDP (horizontal axis) and inflation (vertical axis) for the period 1960-1988. Panel B is the same period, but with the real 5-year rate on the vertical axis. We see a clear relationship between debt and the real interest rate, but no strong feedback to inflation. Conversely, Panels C and D present the same variables, but for the period 1989-2000. Here, we see a strong relationship between debt and inflation, and a weak relationship with real rates. One natural narrative is that the Federal Reserve under Paul Volker was battling inflation at the same time as Reagan was expanding deficits, generating a bond market equilibrium that adjusted primarily (or exclusively) via changes in nominal and real interest rates. On the other hand, while H.W. Bush and Clinton were reducing debt, the Fed under Greenspan kept real rates high, leading to an equilibrium adjustment primarily through lower inflation.

These facts and associated narratives are not presented as a rigorous test of a particular theory. For our purposes, the main takeaway is the non-controversial conclusion that equilibrium adjustments to large changes in government debt stocks work through a combination of prices (inflation and interest rates) and quantities (real income), and the exact mix depends on the policy response. Our contribution is to develop a transparent textbook framework to analyze exactly this interplay and use it to address key policy challenges.

Related Literature

Our work builds on the literature that has integrated the Blanchard-Yaari perpetual youth model into monetary models. Marini and Ploeg (1988) and Cushing (1999) identify monetary non-



(a) Inflation (1960-1988)

(b) Real Rates (1960-1988)

(c) Inflation (1989-2000)

(d) Real Rates (1989-2000)

Figure 2: Debt, Real Rates, and Inflation Across Two Episodes

neutralities in this set-up in the context of flexible prices. Piergallini (2006) and Nistico (2016) study optimal monetary policy in environments with sticky prices and find that strict inflation targeting might no longer be optimal because of the additional financial wealth effects this framework contains. Galí (2021) and Piergallini (2023) focus on the case of low interest rates, $R < G$, and study the implications for asset pricing bubbles and liquidity trap equilibria.[6] Relative to this work, our contribution focuses on the interactions between monetary and fiscal policy, as in the recent work of Angeletos, Lian, and C. K. Wolf (2023). These authors explore how and when fiscal deficits can be "self-financing," either because they generate a boom in output that raises (proportional) taxes, or (with nominal bonds) because they generate inflation. In recent work, Kaplan, Nikolakoudis, and G. Violante (2023) explore the mechanics of the fiscal theory of the price level in a heterogeneous agent economy. Our focus is not on how deficits can be self-financing or the fiscal theory of the price level, but rather on the interplay of fiscal and monetary policy in improving the cross-sectional allocation of output via government bonds. Moreover, we study

[6]See also the contributions of Lepetit (2022), Leith and Wren-Lewis (2000), Nisticò (2012).

global non-linear solutions which contrasts with the first order approximation approach used in these papers.

In related work, Michaillat and Saez (2021) study the global dynamics of New Keynesian models when each household obtains a utility value from its *relative* wealth with respect to the population. They analyze and resolve several of the anomalies present in the standard New Keynesian model, using a phase diagram analysis similar to ours. Their model however features Ricardian equivalence, and thus it does not address the effect of changes in the level of government debt, which is our focus.

Our paper is also related to the growing literature on Heterogeneous Agents New Keynesian (HANK) models, which highlight the interaction between fiscal and monetary policies. In HANK models, government debt-financed transfers are non-Ricardian because richer households, who hold a larger share of the debt, have a lower marginal propensity to consume (MPC) than poorer households. Kaplan, Moll, and G. L. Violante (2018), for example, show that the effectiveness of monetary policy depends crucially on the fiscal response. Specifically, monetary expansions are more potent when the government transfers the savings from the interest rate expense back to households. Auclert, Rognlie, and Straub (2018) study directly fiscal policy multipliers and highlight the importance of the aggregate intertetemporal MPCs, for the magnitude of the multipliers. Other studies, such as McKay, Nakamura, and Steinsson (2016), have examined the impact of forward guidance and the ELB, and found that precautionary savings motives can also temper the power of forward guidance. Fernández-Villaverde et al. (2022) finds that the frequency of ELB events depends on the wealth distribution because the level of interest rates tends to be lower with less and more unequally distributed wealth. Our simpler OLG model shares some of the properties of these richer HANK models as in both frameworks households' wealth matters for the determination of interest rates. Our theoretical analysis complements the existing quantitative work and provides sharp insights that can be useful in other applications.

In our framework, government debt matters for the interest rate directly, beyond its effect on aggregate consumption, because the holdings of government debt differs across generations. This is related to findings in Krishnamurthy and Vissing-Jorgensen (2012), that illustrate empirically that government debt carry a convenience yield, which tends to fall with more debt, and that provide a framework to rationalize these findings, where a representative agent values government bonds them in the utility function. Building on this work, Mian, Straub, and Sufi (2022), study how the restrictions from the ELB on monetary policy interacts with fiscal policy. Similar to our perpetual youth environment, they find that a low level of government debt can increase the likelihood of a binding ELB equilibrium. However, their emphasis is on the impact of this configuration on the fiscal space of the government rather than on inflation, which never exceeds the target level. Finally, in a series of papers Caballero, Farhi, and Gourinchas (2017), Caballero and

Farhi (2017), and Caballero, Farhi, and Gourinchas (2021) explore the role of government bonds in improving economic outcomes when monetary policy is constrained by the ELB. Our analysis of fiscal policy at the ELB echoes their emphasis on how a shortage of government debt can generate slumps at the ELB and how bonds become an aggregate "demand shifter." We extend the analysis to the impact of debt on global dynamics away from the ELB and to the distribution of resources across heterogeneous agents.

2 Environment

The environment builds closely on the canonical perpetual youth model of Blanchard and Yaari, embedded in the textbook New Keynesian paradigm, as in Gali. Time is continuous and there is no aggregate uncertainty. All announcements will be zero probability "MIT" shocks. A measure-one of workers provide labor and save in a government bond. Workers are subject to a constant hazard of death, which they insure via annuities, and have a lifecycle of labor earnings that induces the young to actively save in government bonds. On the production side, firms produce differentiated intermediate goods, compete monopolistically, and face quadratic price adjustment costs as in Rotemberg. Workers and the owners of firms (entrepreneurs) are segmented in the sense that workers cannot own shares in firms. This will be useful to separate the return on government bonds from the internal rate of return to private equity. Finally, the government conducts fiscal and monetary policy. In the following subsections, we fill in the details on each block of the model and then characterize the equilibrium.

2.1 Workers

The worker sector closely follows Blanchard (1985). At any point in time, there is exist a unit measure of workers, a random fraction $\lambda > 0$ of whom will die. At each instant, λ new workers are born. We extend the model to include population growth in Appendix B. The expected lifespan of a worker is therefore $1/\lambda$, and the cross-sectional distribution of age is exponential. In particular, at time t the size (density) of the surviving cohort born at $s \leq t$ is $\lambda e^{-\lambda(t-s)}$.

A (representative) worker born in period s and alive at $t \geq s$ has preferences given by:

$$\int_t^\infty e^{-(\rho+\lambda)(\tau-t)} u(c(s,\tau), n(s,\tau)) d\tau, \tag{1}$$

where ρ is the subjective discount factor; $c(s,t)$ is consumption of the final good; and $n(s,t)$ is the amount of labor supplied. Workers effectively discount the future with the sum of the discount factor and the probability of dying. It will be useful to consider the following functional form:

$$u(c,n) = \ln c + \psi \ln(1-n),$$

with $c \geq 0$ and $n \leq 1$.[7]

Mirroring Blanchard, a worker's productivity z changes over the life cycle. Specifically, z declines exponentially with age at rate $\alpha \geq 0$:

$$z(s,t) = \left[\frac{\alpha+\lambda}{\lambda}\right] e^{-\alpha(t-s)},$$

where the constant in square brackets normalizes the aggregate productivity (summing across cohorts) to 1. In Appendix B, we extend the model to include productivity growth.

As in Blanchard, workers can perfectly insure their survival risk in spot annuity markets. Let $i(t)$ denote the nominal return on government bonds. For each nominal unit ("dollar") held by a worker in the annuity, they receive $(i(t)+\lambda)dt$ if they survive the next $dt \longrightarrow 0$ periods. If they die, the insurance intermediaries receive the asset. As λdt workers die, the insurance sector breaks even with probability one.

Let $P(t)$ be the price of the final good at time t and let $W(t)$ denote the nominal wage per efficiency unit of labor. Workers of cohort s at time t pay non-distortionary taxes $P(t)T(s,t)$. We index taxes by cohort in order to allow the tax burden to decline with productivity. Specifically, let

$$T(s,t) = T(t)z(s,t), \tag{2}$$

where, to economize on notation, we denote the aggregate tax burden as $T(t) \equiv \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} T(s,t) ds$ and the cohort-specific burden as $T(s,t)$.

Let $P(t)a(s,t)$ denote the nominal asset position of the representative agent from cohort s at time t. The flow budget constraint for cohort s is given by:

$$\begin{aligned} \frac{d}{dt}\left[P(t)a(s,t)\right] &= \dot{P}(t)a(s,t) + \dot{a}(s,t)P(t) \\ &= (i(t)+\lambda)P(t)a(s,t) + W(t)z(s,t)n(s,t) - P(t)c(s,t) - P(t)T(s,t), \end{aligned}$$

where a "dot" indicates the derivative with respect to time. Dividing through by $P(t)$, we have

$$\dot{a}(s,t) = (r(t)+\lambda)a(s,t) + w(t)z(s,t)n(s,t) - c(s,t) - T(s,t), \tag{3}$$

[7] Note that we do not restrict $n \geq 0$. A negative n is equivalent to the worker hiring another individual to assist them in "daily living," removing that unit of time from the production sector.

where $w(t) \equiv W(t)/P(t)$ is the real wage, $r(t) \equiv i(t) - \pi(t)$ is the real interest rate, and $\pi(t) \equiv \dot{P}(t)/P(t)$ is the rate of inflation.[8] Households are subject to the natural borrowing limit, $a(s,t) \geq \underline{a}(s,t)$, which, combined with the log preferences, ensures an interior consumption sequence at an optimum. Letting

$$R(t,\tau) \equiv e^{-\int_t^\tau (r(m)+\lambda)dm}$$

we can integrate this constraint forward to obtain:[9]

$$a(s,t) = \int_t^\infty R(t,\tau)\left[c(s,\tau) + T(s,\tau) - w(\tau)z(s,\tau)n(s,\tau)\right] d\tau. \tag{4}$$

Workers are born with zero wealth; that is, $a(s,s) = 0$ for all cohorts s.

Given a sequence of aggregate taxes $T(t)$ and prices $\{w(t), r(t)\}$, a worker born at time s chooses sequences $\{c(s,t), n(s,t)\}_{t \geq s}$ to maximize (1) subject to (4), with $a(s,s) = 0$, as well as the constraints $c(s,t) \geq 0$, $n(s,t) \leq 1$, and $a(s,t) \geq \underline{a}(s,t)$ for all $t \geq s$.[10] The solution to the workers' problem is characterized as follows:

Lemma 1. *The following conditions characterize the optimal consumption and labor plan of a worker born at s evaluated at t ≥ s:*

(i) The Euler equation:

$$\frac{\dot{c}(s,t)}{c(s,t)} = r(t) - \rho; \tag{5}$$

(ii) The static labor-consumption condition:

$$\psi c(s,t) = w(t)z(s,t)\left(1 - n(s,t)\right). \tag{6}$$

[8]For some policy experiments, we may want to consider an unanticipated lump-sum "helicopter" drop of assets (government) bonds to various cohorts at a fixed time t_0. We will be more explicit about this in Section xx, but to streamline the exposition we will suppress this from the notation until needed.

[9]Implicitly, we are using an equilibrium restriction that $r(t)$ is such that each cohorts life-time resources have finite present value. Sufficient for this, given a bounded wage, is $\lim_{t\to\infty} r(t) > -(\alpha + \lambda)$.

[10]In this environment, the natural borrowing limit is

$$\underline{a}(s,t) \equiv -\int_t^\infty R(t,\tau)z(t,\tau)\left(w(t) - T(t)\right) d\tau,$$

which is the present value of maximal labor earnings (i.e., $n = 1$) net of taxes.

(iii) and a consumption function:

$$c(s,t) = \left(\frac{\rho + \lambda}{1+\psi}\right)\left(a(s,t) + h(s,t) - \mathcal{T}(s,t)\right), \tag{7}$$

where

$$h(s,t) \equiv \int_t^\infty R(t,\tau) w(\tau) z(s,\tau) d\tau$$

represents potential "human wealth" and

$$\mathcal{T}(s,t) \equiv \int_t^\infty R(t,\tau) z(s,\tau) T(\tau) d\tau. \tag{8}$$

represents the present value tax burden.

In the next subsection, we discuss aggregation. We flag a few elements of the individual worker's problem that will be useful. One is that all cohorts that are alive have consumption that grows at the same rate $(r(t) - \rho)$ and the level of consumption is linear in total wealth net of taxes. A second feature is that, given wages, labor supply is linear in consumption, with labor income equal to potential income $w(t)z(s,t)$ minus a fraction ψ of consumption. This feature allows us to express consumption as a fraction financial wealth and potential human wealth.

2.1.1 Aggregation

We now characterize the aggregate behavior of the workers, integrating over the various cohorts. Given that the size of a cohort s at time t is $\lambda e^{-\lambda(t-s)}$, and letting capital letters indicate aggregate quantities, we can integrate over s to define the aggregates:

$$\begin{aligned}
C^w(t) &\equiv \int_{-\infty}^t \lambda e^{-\lambda(t-s)} c(s,t) ds \\
N(t) &\equiv \int_{-\infty}^t \lambda e^{-\lambda(t-s)} z(s,t) n(s,t) ds \\
A(t) &\equiv \int_{-\infty}^t \lambda e^{-\lambda(t-s)} a(s,t) ds \\
H(t) &\equiv \int_{-\infty}^t \lambda e^{-\lambda(t-s)} h(s,t) ds \\
\mathcal{T}(t) &\equiv \lambda \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{T}(s,t) ds.
\end{aligned}$$

The following characterizes the aggregate behavior of workers:

Lemma 2. *Given a path $\{w(t), r(t), T(t)\}$, worker optimization implies:*

(i) An aggregate Euler equation:

$$\dot{C}^w(t) = (r(t) - \rho + \alpha)C^w(t) - \frac{(\rho + \lambda)(\alpha + \lambda)}{1 + \psi} A(t); \tag{9}$$

(ii) An aggregate labor supply:

$$\psi C^w(t) = w(t) - w(t)N(t); \tag{10}$$

(iii) An aggregate consumption function:

$$C^w(t) = \left(\frac{\rho + \lambda}{1 + \psi}\right) [A(t) + H(t) - \mathcal{T}(t)]\ ; \tag{11}$$

(iv) and an aggregate evolution of financial wealth:

$$\dot{A}(t) = (r(t) - \rho - \lambda)A(t) + w(t) - T(t) - (\rho + \lambda)(H(t) - \mathcal{T}(t)). \tag{12}$$

The aggregate labor supply and consumption function, conditions (10) and (11), follow immediately from integrating across cohorts their static decisions. The aggregate dynamics of consumption and financial wealth, however, also have to consider that fraction λ of the population is replaced each period. Specifically, a random selection λ dies and is replaced with newborns. Those dying in aggregate have average financial wealth $A(t)$ while those being born have zero. The difference in wealth between those dying and those being born matters for the aggregate Euler equation (9) and the evolution of financial wealth (12). Financial wealth $A(t)$ shows up in (9) because richer agents are replaced by poorer agents; the growth of aggregate consumption is lower with high aggregate wealth. The aggregate Euler equation illustrates that the level of household financial wealth A_t matters for aggregate consumption dynamics, in addition to the interest rate, discount rate, and the age profile of productivity. Note also that Lemma 2 holds for an arbitrary distribution of individual financial assets among surviving cohorts.

Given an initial $A(0)$ and a path for $\{w(t), r(t), T(t)\}_{t=0}^{\infty}$, equations (10), (11), and (12) completely characterize the aggregates of the household sector, $\{A(t), C^w(t), N(t)\}_{t=0}^{\infty}$. From (12), aggregate worker wealth evolves "as if" all cohorts inelastically supply one efficiency unit of labor while at the same time spending an extra ψ on consumption. This reflects that for each individual, any increase in consumption reduces labor income at the linear rate ψ via the income effect on labor supply.

2.2 Entrepreneurs

The technology side of the model is familiar from the standard textbook New Keynesian model.[11] There is a measure-one continuum of entrepreneurs, each of whom operates a firm that produces a unique intermediate input $j \in [0,1]$. Intermediate firm technology is given by $y_j(t) = \ell_j(t)$, where ℓ are efficiency units of labor. Firms hire these units in a competitive labor market at real wage $w(t)$.

Entrepreneurs sell their output to a competitive final goods sector that combines inputs using a constant-elasticity-of-substitution technology:

$$Y(t) = \left(\int_0^1 y_j(t)^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} .$$

Given the constant-returns-to-scale technology and competitive behavior, there is no value added generated by this sector and hence no need to detail ownership of final-good firms. The price index of the final good is given by

$$P(t) = \left(\int_0^1 p_j(t)^{1-\eta} dj \right)^{\frac{1}{1-\eta}} ,$$

where $p_j(t)$ is the price of intermediate $j \in [0,1]$.

For simplicity, we assume entrepreneurs (firms) live forever, although this is not crucial for what follows. They have linear utility and discount at the rate $\hat{\rho}$. Entrepreneurial wealth consists of shares of its firm. Shares in firms are (potentially) traded among entrepreneurs, but as mentioned above are not available to workers. Moreover, entrepreneurs do not hold government bonds. This is stated as an assumption, but is consistent with any equilibrium in which $r(t) < \hat{\rho}$ for all t.

The entrepreneur's consumption/savings problem is given by:

$$\max_{\{c(\tau)\}_{\tau=t}^{\infty}} \int_t^{\infty} e^{-\hat{\rho}(\tau-t)} c(\tau) d\tau \qquad \text{s.t.} \int_t^{\infty} e^{-\hat{\rho}(\tau-t)} c(\tau) d\tau \leq V(t),$$

where V is the value of equity held by the entrepreneur at time t. We already impose in the representative entrepreneur's problem that the internal rate of return to equity is $\hat{\rho}$, which follows from the linearity of preferences and the requirement that consumption be interior in equilibrium. In a symmetric equilibrium, entrepreneurs do not actively trade shares among themselves, and the value of an individual entrepreneurs shares will be equal to the value of their firm, denoted

[11] In particular, this part of the model follows Kaplan, Moll, and G. L. Violante (2018) closely. Our Lemma 3 below reproduces their Lemma 1.

Q.

Entrepreneurs compete monopolistically and choose a sequence of prices to maximize the value of their firm. We restrict attention to symmetric equilibria in which all firms pursue an identical policy.

Intermediate good firms face a nominal friction when setting prices. Let $p(t)$ be the nominal price of an individual variety, where we drop the j index. Intermediate good firms choose the rate of change in their nominal price, $x(t) \equiv \dot{p}(t)/p(t)$ and pay a cost $f(x)Y$, where:

$$f(x) = \begin{cases} \varphi \underline{\pi} x - \frac{\varphi}{2}\underline{\pi}^2 & \text{if } x < \underline{\pi} \\ \frac{\varphi}{2}x^2 & \text{if } x \in [\underline{\pi}, \overline{\pi}] \\ \varphi \overline{\pi} x - \frac{\varphi}{2}\overline{\pi}^2 & \text{if } x > \overline{\pi} \end{cases}$$

The costs of price adjustment are weakly convex, continuous, and continuously differentiable. For intermediate inflation rates $x \in [\underline{\pi}, \overline{\pi}]$, for some $\underline{\pi} < 0 < \overline{\pi}$, adjustment costs are quadratic, as in Rotemberg (1982). For extreme rates of change, costs are linear. In the spirit of Nakamura and Steinsson (2010), this captures that for high inflation environments, price setting is different than at moderate inflation rates. We shall see that this induces a vertical Phillips curve at extreme inflation.

The entrepreneur chooses a path of p via control $x = \dot{p}/p$ to maximize the value of the firm:

$$\begin{aligned} Q(t) = \sup_{\{x(\tau)\}_{\tau \geq t}} & \int_t^{\infty} e^{-\hat{\rho}(\tau - t)} \left[\Pi(p(\tau), \tau) - f(x)Y(\tau)\right] dt \\ \text{subject to: } & \dot{p}(t) = x(t)p(t), \end{aligned}$$

where $\Pi(p, t)$ are the real flow profits gross of adjustment costs of a firm charging price p at time t. Note that we assume the government does not tax (or subsidize) entrepreneurs. This rules out implicit transfers to workers through the taxation of entrepreneurs to pay interest on the debt held by workers.

The solution to the optimal pricing plan generates the following Phillips curve:

Lemma 3. *Let $g_Y(t) \equiv \dot{Y}(t)/Y(t)$ denote the real growth rate; $w^\star = (\eta - 1)/\eta$ denote the flexible price optimal inverse markup; and $\kappa \equiv \eta/\varphi$. In a symmetric equilibrium with a path for aggregate for real wages $\{w(t)\}$, aggregate inflation $\pi(t) \equiv \dot{P}(t)/P(t)$ satisfies:*

$$\dot{\pi}(t) = (\hat{\rho} - g_Y(t))\pi(t) + \kappa \left[w^\star - w(t)\right] \qquad \text{if } \pi(t) \in [\underline{\pi}, \overline{\pi}]; \tag{13}$$

and for $\pi(t) \notin [\underline{\pi}, \overline{\pi}]$, we have:

$$(\hat{\rho} - g_Y(t))\, \underline{\pi} = \kappa \left(w(t) - w^{\star}\right) \qquad \text{if } \pi(t) < \underline{\pi}$$
$$(\hat{\rho} - g_Y(t))\, \overline{\pi} = \kappa \left(w(t) - w^{\star}\right) \qquad \text{if } \pi(t) > \overline{\pi}.$$

For interior inflation, $\pi(t) \in [\underline{\pi}, \overline{\pi}]$, the last term in (13) represents the deviation from the flex-price markup, with a positive value indicating that the markup is higher than the flex-price markup. At the extreme rates of inflation, the real wage is uniquely pinned down for any π, generating a "vertical" Phillips curve in $\pi \times w$ space. Let $\underline{w}$ and $\overline{w}$ denote the low and high inflation real wages, respectively.[12]

One loose end is that the firm always has the option to shut down production. Flow profits are negative if $f(\pi(t)) > 1 - w(t)$. We rule out equilibria that violate this condition. In particular, this rules out equilibria in which inflation explodes in either direction.

Let $C^e(t) \equiv \int_0^1 c_j(t)dj$ denote the consumption of the entrepreneurial sector. Given paths $\{w(t), \hat{\rho}\}_{t=0}^{\infty}$, we say $\{C^e(t), Y(t), V(t), Q(t)\}_{t=0}^{\infty}$ and $\{\pi(t)\}_{t=0}^{\infty}$ that solve the entrepreneurs' problem and satisfy Lemma 3 characterize the entrepreneurs' sector.

2.3 Government

The government sets fiscal and monetary policies under full commitment. Fiscal policy consists of a sequence of non-distortionary taxes, $T(t)$, and real debt $B(t)$, subject to the budget constraint[13]

$$\dot{B}(t) = r(t)B(t) - T(t), \tag{14}$$

We assume the government borrows in real bonds promising a real return to differentiate our analysis from the fiscal theory of the price level. We assume a vanishingly small amount of nominal bonds that carry the nominal rate $i(t)$ in order to ensure the Fisher equation $i = r + \pi$ holds in equilibrium.

Our policy experiments involve a discrete change to the stock of government debt. For example, suppose at time t_0 the government expands government debt by a discrete amount. The fiscal authority rebates the proceeds to workers. Note that this involves a sale of bonds offset by an aggregate transfer of equal amount. We could circumvent this sale-transfer by assuming the government simply "helicopter drops" the new bonds to households. Specifically, let $\xi(s, t)$ denote the cumulative bond transfers to cohort s as of time $t \geq s$. Let $d\xi(s, t) = \xi(s, t) - \lim_{\tau \uparrow t} \xi(s, \tau)$

[12]Specifically, $\underline{w} \equiv \hat{\rho}\underline{\pi}/\kappa - w^{\star}$ and $\overline{w} \equiv \hat{\rho}\overline{\pi}/\kappa - w^{\star}$.

[13]A restriction on fiscal policy is that the present value of taxes cannot exceed the present value of labor earnings when workers work the maximal $n = 1$. This is equivalent to $\underline{a}(t, t) \leq 0$, where $\underline{a}(t, t)$ is the natural borrowing limit of a newborn defined in footnote 10.

denote the amount of new bonds transferred to workers of cohort s at time t. Our experiments assume $\xi(s,t) = 0$ for $t < t_0$, and is constant thereafter. Note that ξ can be negative as well as positive, with negative numbers representing an expropriation of assets. Let

$$d\xi(t) \equiv \int_{-\infty}^{t} d\xi(s,t)ds$$

denote the aggregate change to worker assets.

We then augment (3) to incorporate the helicopter drop to write:

$$da(s,t) = [(r(t) + \lambda)a(s,t) + w(t)z(s,t)n(s,t) - c(s,t) - T(s,t)]\, dt + d\xi(s,t).$$

We augment (14) similarly,

$$dB(t) = [r(t)B(t) - T(t)]\, dt + d\xi(t).$$

As noted already, the evolution of aggregate worker wealth evolves independently of the idiosyncratic distribution of wealth, given a path of prices and taxes (see equation 12). Thus, we focus on the aggregate change in government debt, $d\xi(t_0)$, rather than the specifics of the distribution.

Monetary policy is set via a Taylor rule with a zero inflation target:

$$i(t) = \max\{r_0 + \theta_\pi \pi(t), 0\},$$

where 0 is the ELB on nominal interest rates. We impose the "Taylor principle" $\theta_\pi > 1$ to assure local determinancy. The intercept r_0 is a constant that in the baseline equilibrium corresponds to the level of the real interest rate that is realized when inflation is zero. In the experiments below, however, we consider a monetary policy rule with alternative intercepts that can potentially change in response to shocks or fiscal policies.

2.4 Definition of Equilibrium

Given a path for fiscal and monetary policies $\{B(t), T(t), i(t)\}$, an equilibrium is a path of prices $\{\pi(t) = \dot{P}(t)/P(t)\}$ starting from $P(0) = 1$; real interest rates $\{r(t)\}$ for government bonds; real wages $\{w(t)\}$; firm values $\{Q(t)\}$; and quantities $\{Y(t), C^w(t), C^e(t), N(t), A(t)\}$ such that:

(i) $C^w(t)$, $N(t)$, and $A(t)$ solve workers' problem given w, r, and T;

(ii) $C^e(t)$, $Y(t)$, $Q(t)$, and $\pi(t)$ solve the entrepreneurs' problem given w;

(iii) the bond market clears $A(t) = B(t)$;

(iv) the resource condition is satisfied $C^w(t) + C^e(t) = (1 - f(\pi))\, Y(t) = (1 - f(\pi))\, N(t)$; and

(v) arbitrage between nominal and real returns by households imply that the Fisher equation is satisfied, $i(t) = r(t) + \pi(t)$.

3 Characterizing Equilibrium Dynamics

In this section we introduce a phase diagram that will be the main tool to analyze equilibrium dynamics. We do so for a given level of government debt B_0.

As a first step, we note a possibly surprising aspect of the the equilibrium; namely, $Y = N$ is constant. To see this, recall that the aggregate flow budget constraint for workers implies

$$\begin{aligned} C^w(t) &= w(t)N(t) + rA(t) - T(t) - \dot{A}(t) \\ &= w(t)N(t) + rB(t) - T(t) - \dot{B}(t) \\ &= w(t)N(t), \end{aligned}$$

where the second line uses bond-market clearing $A(t) = B(t)$ and the last equality uses the government's budget constraint (14). The static labor-consumption condition (10) states that $\psi C^w(t) = w(t) - w(t)N(t)$. Hence:

$$\psi w(t)N(t) = w(t) - w(t)N(t) \longrightarrow N(t) = 1/(1+\psi)$$

and

$$(1+\psi)C^w(t) = w(t). \tag{15}$$

In equilibrium, the aggregate labor supply curve is "vertical"; that is, independent of other equilibrium outcomes. This result stems from the balanced growth preferences of workers plus the segmentation of bond markets and taxation. Thus, monetary policy works to change the share of output (net of price adjustment costs) going to labor versus profits, but not the total amount of output produced. All effects on equilibrium outcomes are via re-distribution between workers and entrepreneurs.

Turning to dynamics, we start with the Phillips curve (13). Setting $g_Y = 0$ due to the constant labor supply in equilibrium, for $\pi(t) \in [\underline{\pi}, \overline{\pi}]$, we have

$$\dot{\pi}(t) = \hat{\rho}\pi(t) + \kappa\left[w^\star - w(t)\right]. \tag{16}$$

We postpone discussion of dynamics for $\pi(t) \notin [\underline{\pi}, \overline{\pi}]$ to below.

From the workers side, recall the aggregate Euler equation (9). Substituting the asset market-clearing condition $A(t) = B(t) = B_0$, we obtain

$$(1+\psi)\dot{C}^w(t) = (r(t) - \rho + \alpha)(1+\psi)C^w(t) - (\rho + \lambda)(\alpha + \lambda)B_0.$$

It will be useful to work with the dynamics of the real wage rather than worker consumption, as the former is what appears in the Phillips curve. Substituting for C^w using (15), we have

$$\dot{w}(t) = (r(t) - \rho + \alpha)w(t) - (\alpha + \lambda)(\rho + \lambda)B_0. \tag{17}$$

As B_0 increases, in order to clear the bond market we need some combination of a higher $r(t)$, higher $w(t)$, or lower growth rate of wages $w(t)$.

It will be useful to describe a few properties of a steady state. In steady state we have a simple relationship between debt-to-income B/w and the real interest rate. Setting $\dot{w} = 0$ and rearranging, we have:

$$\frac{B}{w} = \frac{r + \alpha - \rho}{(\rho + \lambda)(\lambda + \alpha)}. \tag{18}$$

The right-hand side is linearly increasing in r. The financial autarky interest rate ($B_0 = 0$) is $\rho - \alpha$, and for higher levels of financial assets we have $r > \rho - \alpha$. In the steady state, as debt relative to labor earnings increases, the real interest rate must also increase. It will be useful to define the steady-state real interest rate associated with the flexible price markup, given B:

$$r^\star(B) \equiv \rho - \alpha + (\rho + \lambda)(\lambda + \alpha)B/w^\star. \tag{19}$$

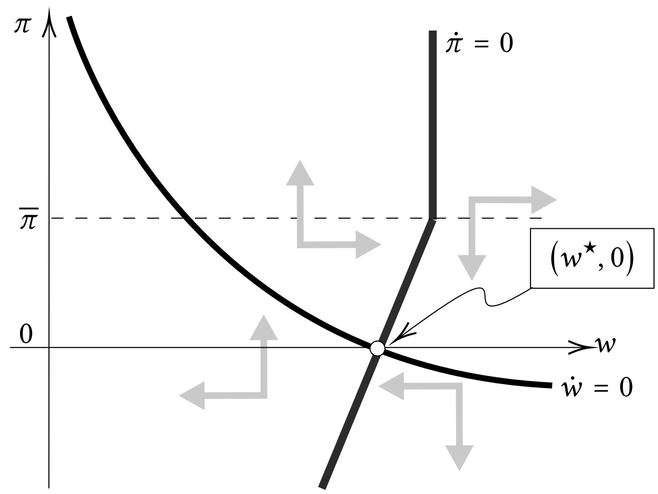
The one remaining element is the real interest rate $r(t)$, which will depend on the monetary policy rule. We first characterize the case in which monetary policy is not bound by the ELB, and then discuss dynamics in the binding-ELB region of the state space.

3.1 Dynamics away from the ELB

When the ELB is not binding, the monetary authority follows the Taylor rule $i(t) = r_0 + \theta_\pi \pi(t)$. Here, we assume the target inflation rate is zero, which requires $r_0 = r^\star(B_0)$. The ELB does not bind for $i(t) > 0$, which requires $\pi(t) > -r_0/\theta_\pi$, which will be the range of inflation relevant for this subsection.

The Taylor rule, combined with the Fisher equation implies $r(t) = i(t) - \pi(t) = r_0 + (\theta_\pi - 1)\pi(t)$.

Figure 3: Phase Diagram: Dynamics Away From the ELB



Substituting into (17), we have

$$\dot{w} = (r_0 + (\theta_\pi - 1)\pi(t) - \rho + \alpha)\, w(t) - (\alpha + \lambda)(\rho + \lambda)B_0. \tag{20}$$

Equations (16) and (20) are two ordinary differential equations (ODEs) in $\pi(t)$ and $w(t)$. This system of two equations, combined with the condition that inflation is bounded and $w \geq 0$ in equilibrium, characterize all possible equilibria in which the ELB does not bind.

To analyze the dynamic system, we use the phase diagram in Figure 3. The curve labelled "$\dot{\pi} = 0$" sets $\dot{\pi}$ in (16) to zero:

$$\pi = \frac{\kappa}{\hat{\rho}}\left(w - w^\star\right) \qquad \text{for } \pi \in [\underline{\pi}, \overline{\pi}]. \tag{21}$$

As w increases above $w^\star$, firms would like to raise their markup. This is counterbalanced by the costs of adjusting prices faster. The stationary point trades off higher π against higher w.

Along the $\dot{\pi} = 0$ curve inflation is constant, at a value that increases with w. Above this locus, $\dot{\pi} > 0$, and below we have $\dot{\pi} < 0$. These dynamics are represented by the arrows pointing up and down in the phase diagram.

Outside of $[\underline{\pi}, \overline{\pi}]$, the lower terms in (13) imply that the Phillips curve is vertical. There is a subtlety when it comes to dynamics for $\pi \notin [\underline{\pi}, \overline{\pi}]$. Along the vertical portion of the $\dot{\pi} = 0$ locus, constant inflation is consistent with profit maximization, but so are movements along the vertical section, as firms are indifferent about the choice of π.

The curve labelled "$\dot{w} = 0$" is the locus of points at which $\dot{w} = 0$. From (20), we have:

$$\pi = -\left(\frac{r_0 - \rho + \alpha}{\theta_\pi - 1}\right) + \left(\frac{(\alpha + \lambda)(\rho + \lambda)}{\theta_\pi - 1}\right)\frac{B_0}{w}. \tag{22}$$

As w increases, the bond market requires a lower real interest rate. Given that $\theta_\pi > 1$, this implies a lower rate of inflation and an even lower nominal interest rate. This generates a negative relationship between w and π in order to keep $\dot{w} = 0$.

From (20), we see that as π increases for a given w relative to the $\dot{w} = 0$ locus, $\dot{w} > 0$. These dynamics are depicted by the horizontal arrows in Figure 3. Note that when $B_0 = 0$ the $\dot{w} = 0$ curve would be horizontal at with inflation equal to the first term in (22).

Again, care must be taken for $\pi \notin [\underline{\pi}, \overline{\pi}]$. For inflation outside this interval, there are unique levels of wage rates, $\underline{w}$ and $\overline{w}$, that are consistent with firm optimization. Thus, there are no wage dynamics in equilibrium outside $[\underline{\pi}, \overline{\pi}]$; that is, if $\pi \notin [\underline{\pi}, \overline{\pi}]$ is part of an equilibrium trajectory, w is constant while the economy moves along the relevant vertical portion of the $\dot{\pi} = 0$ curve.

The intersection of the two curves is the zero-inflation steady state (denoted by $(w^\star, 0)$), which is the target of the monetary policy. This requires the Taylor rule intercept r_0 to be set to $r^\star(B_0)$, so that the bond market clears at the zero-inflation steady state.[14] Note, achieving the zero inflation outcome requires an intercept that depends on the level of debt, which reflects the non-Ricardian environment, and anticipates our discussions of fiscal and monetary policy coordination. The zero inflation steady state is unstable. In particular, the eigenvalues of the linearized system evaluated at the steady state both have real parts strictly greater than zero.

Before using Figure 3 to analyze policy interventions, we discuss dynamics when the ELB binds.

3.2 Dynamics with the ELB

The phase diagram in Figure 3 depicts equilibrium trajectories when the monetary authority is unencumbered by the ELB. As noted above, this will be the case for $\pi(t) \geq -r_0/\theta_\pi$. For $\pi(t) < -r_0/\theta_\pi$, the Taylor rule cannot be followed without running afoul of the ELB.

When the ELB binds, the nominal interest rate is zero, and the Fisher relation implies $r(t) = -\pi(t)$. Substituting this into (17), we have

$$\dot{w} = (-\pi(t) - \rho + \alpha)\, w(t) - (\alpha + \lambda)(\rho + \lambda)B_0. \tag{23}$$

[14] Thus, the plot as drawn requires that $r_0 = r^\star(B_0) > 0$. We will discuss the case where the ELB binds at the zero inflation steady state later in Section 4.3.

Hence, for $\pi(t) < -r_0/\theta_\pi$, the stationary points for w are given by:

$$\pi = -\rho + \alpha - (\alpha + \lambda)(\rho + \lambda)\frac{B_0}{w}. \tag{24}$$

When the ELB binds, we have a positive relationship between π and w that keeps $w(t)$ constant. A lower real interest rate is obtained by higher inflation under the ELB, and hence bond-market clearing requires a higher w for a higher π when $\dot{w} = 0$. Note that equation (23) does not depend on any aspects of monetary policy but is sensitive to the amount of government debt. We return to this in Section 4.3 when we discuss how to use fiscal policy to escape the ELB.

In Figure 4 we add this additional locus to that of Figure 3. The horizontal line labelled *ELB* demarcates the threshold $\pi = -r_0/\theta_\pi$. Note that if $B_0 = 0$, both schedules are independent of w, which would be the case in a Ricardian environment in which the stationary real interest is independent of the level of debt and income.[15]

From (23), we see that as π increases for a given w relative to the $\dot{w} = 0$ locus, $\dot{w} < 0$. Thus, the direction of change for w and π are the same between the two $\dot{w} = 0$ loci, regardless of whether we are above or below the ELB threshold. Hence, there is no discontinuity in trajectories at the ELB threshold.

Figure 4 depicts a situation where there are two stationary points. The first steady state is that depicted already in Figure 3, at which the ELB does not bind. At the other steady state, the ELB is binding, and the dynamics around it are saddle path stable. The existence of this steady state, and the dynamics around it, follows the logic spelled out in Benhabib, Schmitt-Grohé, and Uribe (2001).[16]

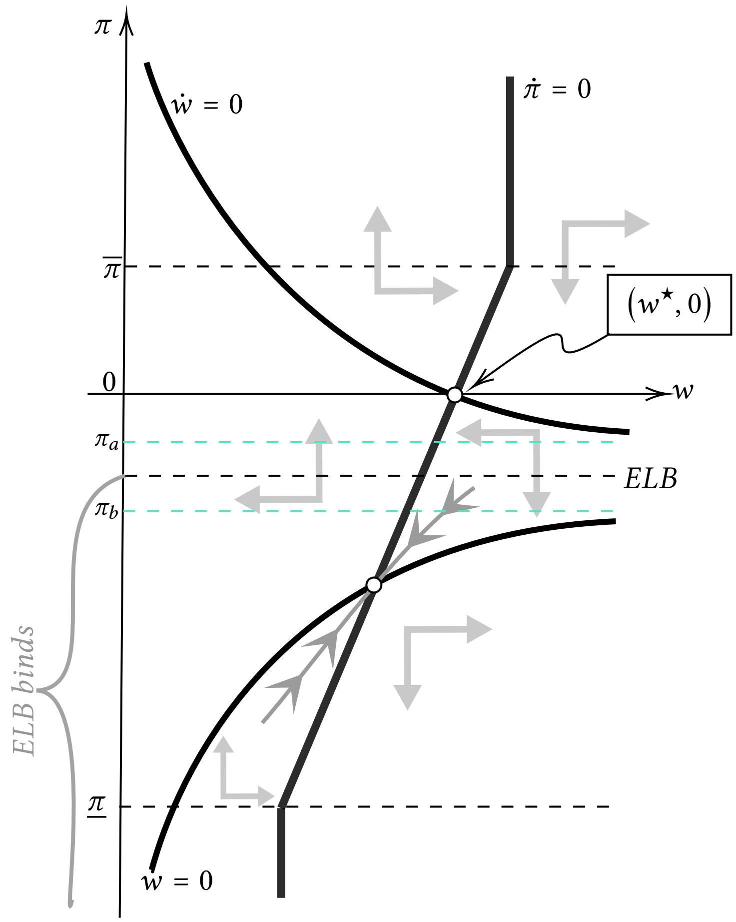
4 Monetary and Fiscal Policy Interactions

With the phase diagram in hand, we can characterize equilibrium dynamics for various scenarios. In particular, we can trace out trajectories for alternative combinations of fiscal and monetary policies. Our four experiments involve (i) an anticipated fiscal deficit/government bond issuance; (ii) Pareto improving debt issuances; (iii) escaping the ELB via fiscal policy; and (iv) the forward guidance puzzle.

[15] The Ricardian environment corresponds to $\alpha = \lambda = 0$.

[16] Note that it is possible to have further steady states at the ELB region. If the lower part of the $\dot{w} = 0$ curve had intersected the $\dot{\pi} = 0$ curve twice, we would have a third steady state, which would be unstable, and if three times (which is possible due to the vertical section of the Phillips curve), then there is an additional stable point, as well. The fact that there could be multiple possible steady states in the ELB region is a consequence of the failure of Ricardian equivalence in the model.

Figure 4: Phase Diagram



Note: ELB represents the value $\pi = -\frac{r_0}{\theta_\pi}$. For inflation levels below this value the ELB constraint is binding; while it is not above. The values of π_a and π_b represent the asymptotes of the respective $\dot{w} = 0$ lines. Their values are $\pi_a = \frac{r_0-\rho+\alpha}{\theta_\pi-1}$ and $\pi_b = \alpha - \rho$. The graph is drawn for the case where $\pi_a > -\frac{r_0}{\theta_\pi} > \pi_b$.

4.1 Anticipated Deficits

Our first analysis involves an anticipated debt-financed tax cut (or transfer increase). Many politicians run on such a platform, including US Presidents Reagan and George W. Bush, as well as, most recently, the mini-budget of UK Prime Minister Elizabeth Truss. We show that anticipated deficits can increase inflation and may increase or decrease wages on impact, depending on the horizon and the response of monetary policy.

We initialize the current period as $t = 0$ and assume we are at the zero inflation steady state with some level of debt B_0 for $t < 0$. At $t = 0$, there is an unanticipated announcement that at time $t' > 0$ the government will increase debt to $B' > B_0$ and rebate the proceeds to workers. We trace out the path of inflation and real wages until the economy reaches the new steady state. In doing so, we focus on the dynamics away from the ELB.

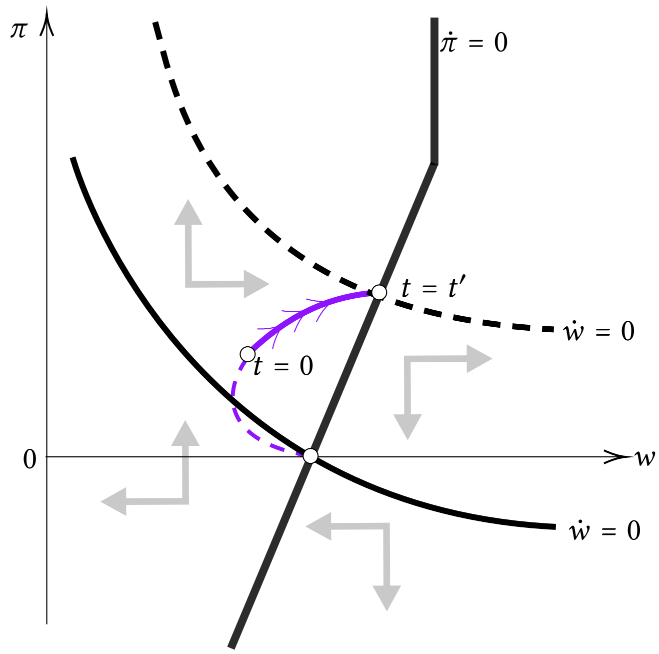
In Figure 5, we replicate the phase diagram from Figure 3. At t', the "$\dot{w} = 0$" shifts out due to the change in B, which is the dashed line labelled $\dot{w} = 0$. The size of this shift for a given B' depends on parameters, in particular λ, ρ, and α, with the Ricardian case of $\alpha = \lambda = 0$ involving no shift. Prior to t', the economy is still subject to the dynamics governed by the original B_0, which are depicted by solid lines and the arrows. The Phillips curve does not depend on B, and hence the "$\dot{\pi} = 0$" remains stable.

The economy jumps to the trajectory at $t = 0$ and travels along that path until it reaches the new steady state at exactly $t = t'$. For larger t', the announcement effect places the economy closer to the original steady state; for smaller t', the economy jumps closer to the eventual steady state. Depending on parameters, the eigenvalues may be complex or real, and the resultant path may cycle or not, respectively.

We depict a thick solid portion of the trajectory as an example path. The initial point involves higher inflation and lower real wages. This is combined with positive $\dot{w}$ and $\dot{\pi}$. The economy's response can be understood through the logic of the bond market (which is the flip side of the goods market). A higher eventual wage (and hence $\dot{w} > 0$) lowers demand for the initial (fixed) stock of bonds for standard inter-temporal substitution reasons. The bond market clears at $t = 0$ with a higher real interest rate, which raises the demand for bonds. The higher real interest rate must be accompanied by higher inflation, due to the monetary authorities Taylor rule: $r = r_0 + (\theta_\pi - 1)\pi$, with $\theta_\pi > 1$. Whether the jump in π is associated with an increase or decrease in the $t = 0$ real wage depends on the time horizon and the parameters of the model.

For the experiment of Figure 5, we held the intercept r_0 constant in the Taylor rule at $r^\star(B_0)$. The increased B requires an increase in the real interest rate, but with a stable Taylor rule this must be associated with higher inflation. If the monetary authority increased its intercept at t', the shift in the $\dot{w} = 0$ curve would be dampened. In the limit, the central bank can keep the economy

Figure 5: Response to an Anticipated Bond Issue



at the zero inflation initial steady state for all $t > 0$ by promising to increase the intercept (and hence the nominal interest rate) one-for-one with the necessary increase in the real interest rate to absorb the new bonds at the initial level of income. In particular, this outcome is possible by setting $r_0 = r^\star(B')$ for $t \geq t'$.

The primary conclusion from this analysis is that in a non-Ricardian environment, the economic consequences of anticipated deficits hinge on whether the monetary authority follows a set rule or *changes* its feedback rule. If the central bank follows the policy advice obtained from the Ricardian benchmark that the long run target real interest rate is invariant to fiscal policy, the consequences are higher inflation and avoidable fluctuations in incomes. To an outsider well versed in the Ricardian literature, the central bank appears to be doing exactly as prescribed; namely, following a set monetary rule that leans against lax fiscal policy. However, the rigidity of the rule is cause rather than cure for the inflation observed in equilibrium.

4.2 A (Robust) Pareto Improvement: Channeling Samuelson

Our second exercise considers whether and how increasing the stock of government debt affects the distribution of consumption and welfare in our New Keynesian environment with heterogeneous agents.

We begin with the fact that the only aggregate store of value for workers in this economy are government bonds. A long literature based on the Bewley model (Bewley, 1979; Bewley, 1983;

Huggett, 1993; Aiyagari, 1994) has studied the welfare consequences of government debt, starting with the seminal contributions of Woodford (1990) and Aiyagari and McGrattan (1998). Using a real model with capital based on Aiyagari (1994), in Aguiar, Amador, and Arellano (2022) we showed that when r is less than the growth rate there may be scope for a "robust" Pareto improvement (RPI) via a government bond issuance. We defined an RPI to be a policy that induced a change in prices and taxes such that the budget set of any agent was guaranteed to be weakly expanded at any state and time. The advantage of this criteria is that it ensures a Pareto improvement regardless of how agents trade off inter-temporally (or across states, with uncertainty). In an RPI, with a weakly greater flow income at every date, the initial equilibrium consumption path for any agent remains affordable at the initial labor supply, and hence, their welfare cannot fall and must increase if the budget set strictly expands. The RPI will also serve as a useful starting point to discuss the full class of Pareto improvements. We show achieving a Pareto improvement requires coordination of monetary and fiscal policies. As the primary mechanism to an improvement exploits the lack of Ricardian equivalence, the lessons learned may be extendable to the now-popular quantitative heterogeneous agent New Keynesian models (HANK).

Robust Pareto Improvements. Suppose we start with an initial stationary equilibrium, which we indicate by superscript "o". That is, wages are w^o, inflation is π^o, etc. We will consider a simple policy change: at $t = 0$, the government changes the total stock of debt to B' from $B^o \geq 0$ and adjusts taxes on workers to satisfy its budget constraint.[17] Moreover, the monetary authority adjusts its Taylor rule. After $t = 0$ there are no additional policy changes, debt remains constant at B', and taxes are adjusted to satisfy the government budget constraint. Thus, the economy jumps to a new stationary point, which we denote by a prime superscript.

The question we ask is whether there is a combination of monetary and fiscal policies such that no worker or entrepreneur has a reduction in income at any point in time, and at least one agent has a strict increase. If so, this is an RPI, and hence a Pareto improvement.

Note that at the moment of the policy change, the government must transfer an additional $B' - B^o$ of resources (or tax, if negative) to workers. To make sure that we have an RPI, we require then that $B' > B^o$. The flow income of a worker of cohort s in subsequent periods, after the policy change, is :

$$(r' + \lambda)a + w'z(s,t)n - z(s,t)T'$$

where a and n represent the asset and labor supply choices for cohort s at t; and r', w' and T' represent the new prices and tax levels. The flow income of an entrepreneur is profits minus price

[17] As noted above, in case of a bond issuances, this is equivalent to the government distributing bonds directly to the existing worker cohorts, increasing aggregate savings by $B' - B^o$.

adjustment costs:

$$\Pi' - f(\pi') = (1 - w' - f(\pi'))\, Y' = \hat{\rho} Q'.$$

where π' and Y' represent the new inflation and output levels.

An RPI is generated if $Q' \geq Q^o$, $T' \leq T^o$ and, for any (s,t), $r'a^o(s,t) \geq r^o a^o(s,t)$, $w'n^o(s,t) \geq w^o n^o(s,t)$, where $a^o(s,t)$ and $n^o(s,t)$ are the original equilibrium choices of assets and labor supply for cohort s at time t. If the new equilibrium satisfies these conditions, given that the initial transfer to workers is strictly positive ($B' > B^o$), then the original consumption and labor supply decisions remain feasible for workers after the policy change (so their welfare cannot fall, and has strictly increase for the generations alive at $t = 0$), and the welfare of the entrepreneurs has weakly increased.[18] This is then sufficient for a Pareto improvement.

Recall from Section 3 that Y^o is constant. Assuming that $\pi^o \in (\underline{\pi}, \overline{\pi})$, for $w' \geq w^o$ it is necessary that $\pi' \geq \pi^o$ given the stationary Phillips curve. To ensure $Q' \geq Q^o$, it suffices that $w' = w^o$ and $\pi' = \pi^o$, conditions that we impose in what follows.[19] From the government's budget constraint, $T' = r'B'$. Recall that cohort s has a tax burden $z(s,t)$ times the aggregate tax, and thus every workers' tax burden strictly increases in T'. A requirement for an RPI is that $T' \leq T^o$, which in turn means $r'B' \leq r^o B^o$. Given that $w' = w^o$ and $B' > B^o$, it follows from the steady state relationship between debt-to-income and real rates, (18), that $r' > r^o$. The final element of the workers' flow income to check is that $r'a^o(s,t) > r^o a^o(s,t)$. We will show that no cohort has a negative asset position in the initial equilibrium, and thus this last condition is satisfied with an increase in r.[20]

Both $r'B' \leq r^o B^o$ and $r' > r^o$ with $B' > B^o \geq 0$ can only be possible if $0 \geq r' > r^o$. That is, we start in an equilibrium with negative real rates (which is possible if $\rho < \alpha$). Using (18), rB is weakly decreasing in B if $r^o \leq (\rho - \alpha)/2 < 0$, where $(\rho - \alpha)/2$ represents the peak of $-rB$ curve (the peak of the transfer Laffer curve). An RPI may go beyond the peak of this curve, but the initial equilibrium needs to be on the downward sloping portion.

The final question concerns the adjustments that the monetary authority most do to its policy rule to guarantee that the RPI is an equilibrium outcome. Given that $\pi^o = i^o - r^o$, and $i^o \geq 0$, an RPI requires an initial equilibrium with strictly positive inflation.[21] After the policy change, the

[18]For the definition of an RPI see Aguiar, Amador, and Arellano (2022).

[19]An alternative would be to start from a deflationary environment and have π increase but $|\pi|$ decrease, which could increase net profits even with an increase in w. As we shall see, having $\pi^o < 0$ will run afoul of the ELB.

[20]Every cohort has a consumption profile that changes at a constant rate, $r^o - \rho$, and a labor earnings profile (net of taxes) that changes at a constant but weakly lower rate, $-\alpha \leq r^0 - \rho$. The latter is a requirement for the economy to sustain non-negative levels of government debt in the stationary equilibrium (see (18)). Given that newborns have zero assets, it follows that the asset holdings of every cohort must remain non-negative over their lifecycle. See the proof of Lemma 5 for a detailed derivation that $a(s,t) \geq 0$ for all s and t.

[21]Part of the reason for this necessary result is the absence of economic growth in the model. If there were some baseline population or technological growth, $n + g$, then the condition should be related to $r^o < n + g$. In this case,

monetary policy needs to choose a new intercept for its Taylor rule, r'_0, that is consistent with the higher real interest rate. In particular, the rule is $i(t) = r' + \pi^o + \theta_\pi(\pi(t) - \pi^o)$, where π^o is the target inflation and the intercept is $r' + \pi^o$. Given that $i = r' + \pi^o > r^o + \pi^o = i^o \geq 0$, it follows that the new stationary equilibrium will necessarily be away from the ELB.

We summarize the above discussion with the following lemma:

Lemma 4 (Existence of an RPI)**.** *Suppose the economy is in an initial stationary equilibrium with $r^o \leq (\rho - \alpha)/2 < 0$. Then there exists a new equilibrium with a combination of fiscal and monetary policies that constitutes a (robust) Pareto improvement.*

Note that while agents have more flow income in the aggregate, there is no change in total output. The increase in the interest rate needed to clear the bond market ensures that aggregate consumption does not change. The welfare improvement comes from a better distribution across cohorts of the same amount of aggregate income.

We highlight here that an RPI arises from a simple policy change: an increase in debt, a corresponding reduction in taxes, and an adjustment to the monetary policy rule. For example, we do not require sophisticated changes in the tax system: the spirit of the RPI approach is that very little information about how agents value trade-offs is required to engineer a Pareto improvement. In the simple environment we are analyzing here, we can exploit our knowledge of the agents preferences (something that we will do next). But in more complex environments, with richer agent heterogeneity, such welfare comparisons may become much harder. Note one more thing: Lemma 4 shows that an RPI is easy to guarantee if the initial equilibrium real rate is low enough. This result is special to the fact that our current model has no capital. The presence of capital accumulation may require changes in subsidies to guarantee that capital is not crowded out with an increase in B. These additional fiscal costs may need to be taken into account when evaluating the feasibility of an RPI. To see how an RPI can be used in a real model with richer agent heterogeneity and with capital accumulation, we refer the reader to Aguiar, Amador, and Arellano (2022).[22]

Pareto Improvements. The RPI result relies on a relaxation of all the agents flow budget constraints, guaranteeing that their welfare is weakly higher after the policy change. An alternative approach for evaluating the feasibility of Pareto improving policies is to directly evaluate welfare, and such an approach can also provide more intuition on the nature of the improvement.

a positive nominal interest rate i^o no longer implies strictly positive inflation, as r can be positive if $n + g > 0$. See Appendix B.

[22] In that paper, we showed that, in the presence of capital accumulation, the elasticity of aggregate household savings to the interest rate was a key component for the feasibility of an RPI. We suspect that a similar result will occur in this environment if we were to introduce capital.

Consider how a worker's welfare varies with the real interest rate (and the wage) in a stationary equilibrium. In particular, consider a newborn worker in a such a stationary environment:

Lemma 5. *In a stationary equilibrium with real interest rate r and real wage w, a newborn worker has utility*

$$U^0(r, w) = \frac{(1+\psi)\ln(\rho+\lambda-r)}{(\rho+\lambda)} + \frac{(1+\psi)(r-\rho)}{(\rho+\lambda)^2} + \kappa_0 \ln w + \kappa_1,$$

where $\kappa_0 > 0$ and κ_1 are combinations of parameters.

One can see that the newborn's welfare is increasing in the wage, which represents the share of output going to workers. Their welfare also depends non-monotonically on r. A greater r implies lower discounted lifetime wealth, lowering period t consumption, but a faster growth rate of consumption going forward. But note that U^0 is strictly concave in r for $r < \rho + \lambda$. An immediate implication of Lemma 5 is:

Corollary 1. *For a given wage, the welfare of a newborn worker in a stationary equilibrium is maximized when the real interest rate is zero.*

To see the intuition for this result, first note that the social return to saving is zero. That is, there is no storage technology that yields any other return, and payments on government bonds are simply transfers from tax payers to bond holders. However, individual workers may see a net return to bonds different than zero due to the failure of Ricardian equivalence. From their Euler equation, an $r \neq 0$ distorts the inter-temporal path of consumption relative to the social optimum, which in turn distorts the cross-sectional allocation of the fixed amount of resources. To see this most starkly, consider a planning problem that allocates a fixed amount of resources every period across generations. This planning problem yields the same allocation as that obtained from the competitive equilibrium when there is zero return to saving.[23]

In the appendix, we generalize this result to include technological and population growth at rates g and n, respectively. In that case, the optimal $r = n + g$. Recall that the individual Euler equation sets $\dot{c}/c = r - \rho$. When $\rho = 0$, so there is no subjective time discounting, the optimal consumption trajectory grows at $g + n$, the growth rate of the economy. Population growth on its own leads to increasing consumption over time for an individual due to the Ponzi game nature

[23] Specifically, let τ indicate age and consider the cross-sectional sharing rule:

$$\max_{\{c(\tau)\}} \int_0^\infty e^{-(\rho+\lambda)\tau} \ln(c(\tau))d\tau \quad \text{s.t. } \lambda \int_0^\infty e^{-\lambda\tau} c(\tau) \leq C,$$

where the objective is the expected present value utility of a newborn that consumes according to the age-dependent sharing rule, and the resource constraint weights each age/cohort by their size. This problem is isomorphic to the inter-temporal problem of a newborn individual facing an interest rate of zero.

of inter-generational transfers afforded by government bonds. This was a seminal insight of Samuelson (1958).

Newborn welfare is maximized at $r = 0$, but a Pareto improvement must also account for existing cohorts at the time of the policy change. If $B^o = 0$, then existing cohorts have zero wealth and thus identical to newborns in regard to the welfare consequences of a change in r, but in addition get the initial distribution $B' - B^o$. However, if $B^o > 0$, then existing cohorts have positive wealth and are thus sensitive to changes in r beyond the effects on newborn cohorts. The positive savings gives them an extra benefit from higher real interest rates.

As derived in the proof of Lemma 5 (equation 33), the utility of workers of cohort s at t in a stationary equilibrium with prices r and w satisfies

$$\left(\frac{\rho+\lambda}{1+\psi}\right)U(s,t) = \ln\left(\frac{a(s,t)\lambda(\rho+\lambda)}{w} + (\rho+\lambda-r)e^{-\alpha(t-s)}\right) + \kappa_0 \ln w + \frac{r-\rho}{\rho+\lambda} + \kappa_2,$$

where κ_2 is a combination of parameters. For a given level of $a(s,t) \geq 0$, utility is strictly increasing in the interest rate for $r < 0$. Contrary to newborns, if $a > 0$ utility peaks at r strictly greater than zero, due to the presence of positive assets. This implies the feasibility of Pareto improving policies when $r^o < 0$. These Pareto improvements, as with the RPI discussion above, require a combination of fiscal and monetary policies, because as in Lemma 4, their coordination is necessary to guarantee that the utility of entrepreneurs and workers weakly increase. In particular, monetary policy needs to be adjusted to deliver unchanged inflation π^0 which guarantees unchanged entrepreneur's welfare. The following Lemma summarizes this result.

Lemma 6. *Suppose the economy is in an initial stationary equilibrium with $r^0 < 0$. Then there exists a new equilibrium with a combination of fiscal and monetary policies that is a Pareto improvement.*

In the competitive equilibrium of our economy, many combinations of r and B/w are sustainable as indicated with in schedule (18). For $r \neq 0$, private agents perceive a return on savings that differs from the true social return. This is an artifact of how government bonds are perceived as wealth in a non-Ricardian environment. For $r < 0$, an increase in B increases real rates which benefits all generations in an equilibrium without private borrowing. This is not enough for a Pareto improvement, because absent changes in monetary policy, an increase in B also generates inflation as illustrated in Figure 5, which lowers the entrepreneurs' welfare. A change in monetary policy that increases i one for one with the increase in real rates, however, is able to maintain a stable inflation and therefore keeps entrepreneurs' welfare unchanged. Note that this result holds whether or not the initial equilibrium is at the ELB. If the initial equilibrium is at the ELB the Pareto improvement also takes the economy away from the ELB. In the next subsection, we explore in detail these dynamics.

This subsection extends the insight of Samuelson to a New Keynesian environment.[24] The crucial lesson learned here is that expanding the stock of safe assets is not sufficient to improve welfare. In the New Keynesian model, this will have potentially unappealing inflationary and distributional consequences. However, the pairing of debt issuance with an appropriate shift in the monetary rule can rescue the traditional insight that increasing the quantity of safe assets may improve the distribution of a fixed amount of income. As noted above, this insight has obvious implications for the broader HANK literature.

4.3 Discount Factor Shocks and Avoiding the ELB

In standard New Keynesian models, a transitory increase in patience can generate a deflationary recession and perhaps a binding ELB.[25] These typically stand in for exogenous declines in consumer demand. We now explore such shocks in our framework, map out the dynamics, and show how fiscal policy can avoid the ELB.

Let workers' discount rate be denoted $\rho(t)$, which takes on two values $0 < \underline{\rho} < \overline{\rho}$ and for some $t' > 0$ follows:

$$\rho(t) = \begin{cases} \overline{\rho} & \text{for } t \notin [0, t') \\ \underline{\rho} & \text{for } t \in [0, t'). \end{cases}$$

That is, for $t < 0$, $\rho(t) = \overline{\rho}$. At $t = 0$, $\rho(t)$ unexpectedly declines to $\underline{\rho} < \overline{\rho}$. For $t \in [0, t')$, $\rho(t) = \underline{\rho}$. For $t \geq t'$, the discount rate returns to $\rho(t) = \overline{\rho}$. The change at $t = 0$ is unexpected, but the equilibrium path is one of perfect foresight thereafter.

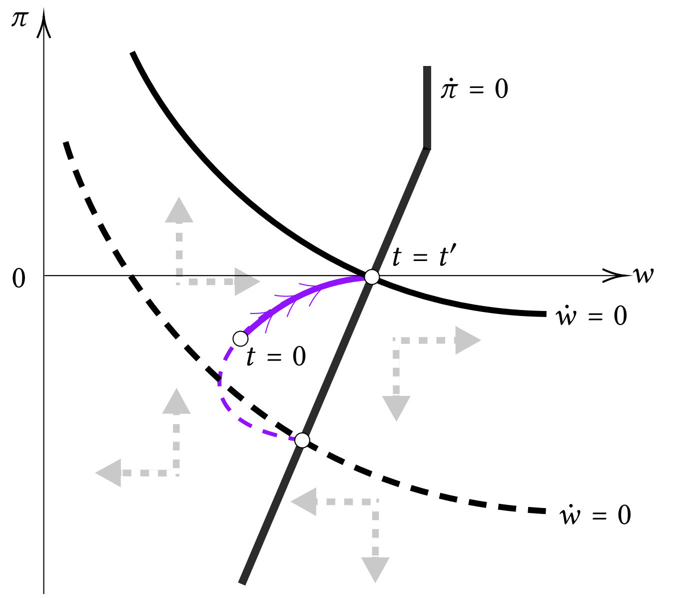
We consider two scenarios, both of which assume that we are at the zero-inflation steady state for $t < 0$. The first case studies a mild increase in patience, such that the zero-inflation steady state remains viable in equilibrium, conditional on an appropriate response of monetary policy. The second is a sharper decline in consumer demand, such that the zero-inflation steady state is no longer viable, absent a response of fiscal policy. In all exercises, we hold the discount rate ($\hat{\rho}$) of the entrepreneurs constant.

We explore the mild scenario in Figure 6. The figure depicts the region of the state space for which the ELB does not bind. The initial (and final) steady state are depicted by the intersection

[24] Balasko and Shell (1980) extended Samuelson (1958) to provide precise conditions on the existence of Pareto improvements in OLG models. In the stationary context, Balasko and Shell show that an equilibrium is Pareto efficient if and only if $r \geq 0$. This subsection extends that result to the standard New Keynesian framework, including the necessary adjustment of the monetary policy rule. Moreover, these improvements can be made with simple policies that do not rely on complex changes to the tax schedule.

[25] For example, see Krugman (1998), Eggertsson and Woodford (2003), and Jung, Teranishi, and Watanabe (2005).

Figure 6: Temporary Decline in ρ: ELB does not bind



of the solid "$\dot{w} = 0$" and "$\dot{\pi} = 0$" lines. This traces out equation (22), setting $\rho = \overline{\rho}$.

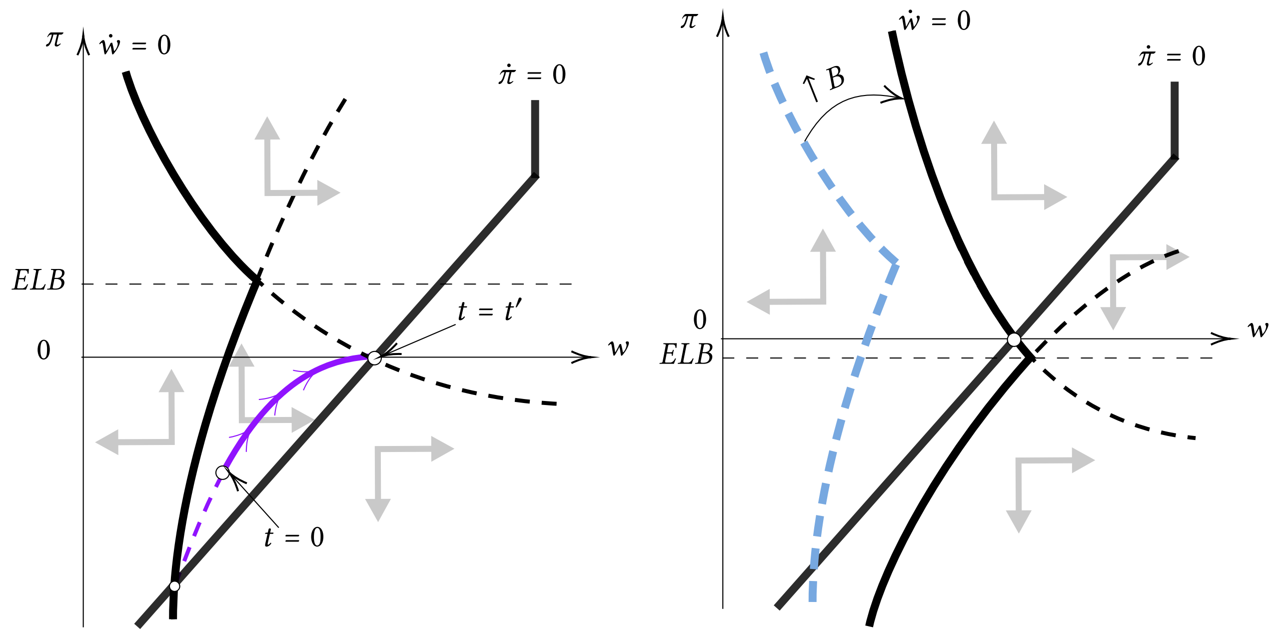
The dashed $\dot{w} = 0$ locus is drawn for $\rho = \underline{\rho}$. From (22), a decline in ρ shifts the locus down in $\pi \times w$ space. The Phillips curve does not shift, as we maintain $\hat{\rho}$ constant in this scenario. The figure depicts the case in which the Taylor rule is held constant, a point we return to below.

The dashed arrows depict the dynamics that hold for $t \in [0, t')$; that is, relative to the dashed locus. Perfect foresight and worker and firm optimization implies that the equilibrium must be back at the initial steady state at $t = t'$. The purple line depicts trajectories that depart and return to that steady state. The precise point on the trajectory that holds at $t = 0$ depends on t'.

As in the case of Figure 5, the potential trajectories correspond to combinations that clear the bond market. In this experiment, B_0 is held fixed, but, all else equal, more patient workers desire to hold more bonds at a given real wage and interest rate. The market clears by a combination of lower w and higher $\dot{w}$, both of which reduced demand. The counterpart is a change in the path of π that ensures the wage dynamics are consistent with firm optimization, which corresponds to lower π and higher $\dot{\pi}$. The inflation path in turn generates a path of the real interest rate via the Taylor rule. A temporary decline in the discount rate, therefore, generates the familiar temporary declines in inflation and labor income.

The monetary authority, nevertheless, could keep the equilibrium at the desired zero-inflation steady state by having r_0 move one-to-one with $\rho(t)$, as was the case in the experiment in Figure 5. In particular, a transitory decrease in the nominal interest rate that matches the decline in ρ maintains bond and good market equilibrium at the initial allocation. Specifically, let $r_0(t)$ in the

Figure 7: Temporary Decline in ρ: Binding ELB and Fiscal Policy



(a) ELB Binds

(b) Increase in B

Taylor rule be adjusted as follows:

$$r_0(t) = \rho(t) - \alpha + (\rho(t) + \lambda)(\lambda + \alpha)B_0/w^\star. \tag{25}$$

From (18), this modification ensures that the nominal interest adjusts so that at zero inflation the bond market clears at $w = w^\star$. Hence, the economics of a moderate temporary decline in the discount rate track that of the textbook Ricardian model. Interestingly, it also echoes the dynamics of the anticipated deficit scenario above, but shifted in $\pi \times w$ space relative to the initial steady state.

A perhaps a more interesting scenario is one in which the ELB prevents the monetary authority from fully accommodating the decline in ρ via a decline in the nominal interest rate. This will be the case if $\underline{\rho}$ is sufficiently low such that $r_0(t)$ defined in (25) is negative when $\rho(t) = \underline{\rho}$. In this case, the ELB threshold is at positive inflation.

This case is depicted in Figure 7 panel (a). We omit the initial (and final) $\dot{w} = 0$ curve, but it intersects the $\dot{\pi} = 0$ at the horizontal axis with the ELB at negative inflation. The downward and upward sloping $\dot{w} = 0$ lines represent equations (22) and (24), respectively, with ρ set to $\underline{\rho}$. The "ELB" line is the inflation threshold at which $i = 0$ (i.e., $\pi = -r_0/\theta_\pi$) under the time-varying Taylor rule with intercept (25) for $t \in [0, t')$, which is positive in this case of when $\rho = \underline{\rho}$.

The sharp decline in ρ makes the zero inflation steady state not achievable for $t \in [0, t')$. The decline in consumer demand is so severe that it reduces interest rates below the ELB point. The dynamics for $t \in [0, t')$ are governed by the upward sloping "$\dot{w} = 0$" locus and the intersection of this curve with the $\dot{\pi} = 0$ Phillips curve is an unstable steady state (see footnote 16). The trajectory that leads away from this intersection and reaches the zero-inflation steady state at $t = t'$ is the equilibrium.[26] This trajectory follows the logic set out in Werning (2011) in his "no-commitment" policy, and echoes that of Krugman (1998), Eggertsson and Woodford (2003), and Jung, Teranishi, and Watanabe (2005). There is initially a large decline in the real wage and inflation, and the economy slowly recovers anticipating a return to "normal" at $t = t'$.

Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), and Werning (2011) argue that commitment to low interest rates after the ELB no longer binds can improve upon the no-commitment policy. Commitment to this post-t' inflationary policy "pulls up" the trajectory: at time t', the equilibrium has higher inflation and income than the zero-inflation steady state, and via anticipation the entire trajectory adjusts. This logic underlies the now-standard "forward guidance" policy prescription when at the ELB.

In our environment, fiscal policy can provide an alternative solution, and one that does not require the same type of commitment as forward guidance.[27] Recall that $r_0(t)$ as defined in (25) is defined for a particular B_0. An increase in B_0 increases r_0, and this shifts the upward sloping portion of the "$\dot{w} = 0$" locus down. For modest increases in B_0, the intersection anchoring the trajectory in Figure 7 moves closer to the zero-inflation steady state. For large enough increases, the curve shifts enough that the ELB no longer binds, which occurs when the associated r_0 is no longer negative. This case is shown in Figure 7 panel (b): fiscal policy is sufficiently large to render the ELB irrelevant, and the monetary authority can implement its target of $w = w^\star$ and $\pi = 0$. Note again the need for monetary and fiscal policy coordination as after the fiscal expansion, the monetary authority must adjust its Taylor rule as the underlying permanent real rate at zero inflation has increased.

It could be the case that the ELB binds indefinitely in our non-Ricardian environment for a given set of parameters; that is, $r^\star(B_0) < 0$. If the monetary authority targeted the zero inflation steady state by setting $r_0 = r^\star(B_0)$, in Figure 7 the economy would be at the intersection of the $\dot{\pi} = 0$ line and the lower $\dot{w} = 0$, and would have permanent deflation and $w < w^\star$. The monetary authority has the option to avoid this by raising the intercept of its Taylor rule, setting $r_0 > 0 > r^\star(B_0)$. This would shift the upper $\dot{w} = 0$ locus out, and the ELB threshold down, creating the opportunity for a positive inflation steady state with $w > w^\star$. At this steady state, $i > 0$ and the ELB would not bind, but inflation and real wages are higher than the original target

[26]Here, we ignore the possibility that there are additional deflationary steady states, which may be possible given the (omitted) vertical portion of the Phillips curve.

[27]Correia et al. (2013), Mian, Straub, and Sufi (2022), and C. Wolf (2021) also discuss dealing with the ELB using alternative fiscal policy tools.

of $w = w^\star$ and $\pi = 0$. Fiscal policy provides a path to the zero inflation steady state. An increase in government debt can increase the long-run real interest rate to the point that the zero-inflation steady state is attainable with $i > 0$.

4.4 The Forward Guidance Puzzle

The phase diagram is also a transparent analytical tool to understand the "forward guidance puzzle" of Del Negro, Giannoni, and Patterson (2023) and McKay, Nakamura, and Steinsson (2016). In a standard representative agent New Keynesian model, Del Negro, Giannoni, and Patterson (2023) showed that an announcement to temporarily reduce real interest rates at some point in the future had a large effect on consumption and inflation in the announcement period, and all periods leading up to the interest rate cut. Moreover, the further in the future the reduction would happen, the larger the initial effect. Using alternative quantitative models, Del Negro, Giannoni, and Patterson (2023), McKay, Nakamura, and Steinsson (2016), and Kaplan, Moll, and G. L. Violante (2018) showed that breaking Ricardian equivalence mitigates this puzzle. Del Negro, Giannoni, and Patterson (2023) builds on Blanchard-Yaari, and hence is closest to our framework. The other two papers build on Aiyagari (1994), but draw similar lessons.

We can adapt the McKay, Nakamura, and Steinsson (2016) experiment to our setting as follows. Let r_0 be the stationary real interest rate $r^\star(B_0)$ that clears the bond market at zero inflation (equation 19). At $t = 0$, the monetary authority announces that at $t_0 > 0$ it will reduce the *real* interest rate below r_0 for the interval $[t_0, t_1)$, before resuming its original policy. This is equivalent to a Taylor rule with $\theta_\pi = 1$ and an intercept that shifts down for $t \in [t_0, t_1)$. Specifically, for some $\Delta > 0$, monetary policy follows:

$$i(t) = \begin{cases} r_0 + \pi(t) & \text{for } t \notin [t_0, t_1) \\ r_0 - \Delta + \pi(t) & \text{for } t \in [t_0, t_1). \end{cases} \tag{26}$$

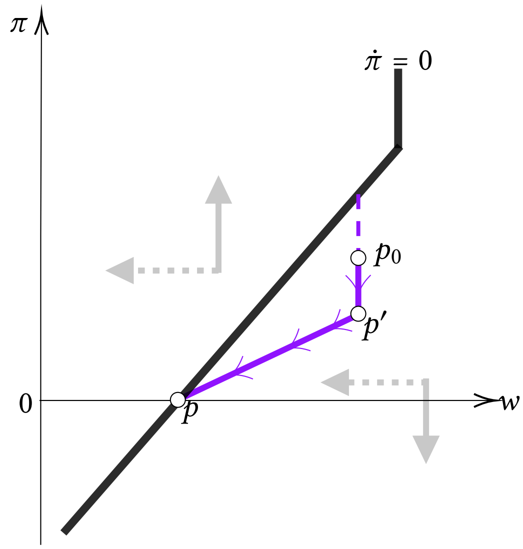
This policy means that for $t < t_1$, the monetary authority is implementing a particular path of the real interest rate. After t_1, we assume that the economy is back at the zero inflation steady state.

To replicate the representative agent "puzzle," consider the case with zero government bonds. From (9), we recover an aggregate Euler Equation that is identical to that of a representative agent economy. The stationary real interest rate is $\rho - \alpha$, which for the current experiment will be r_0.

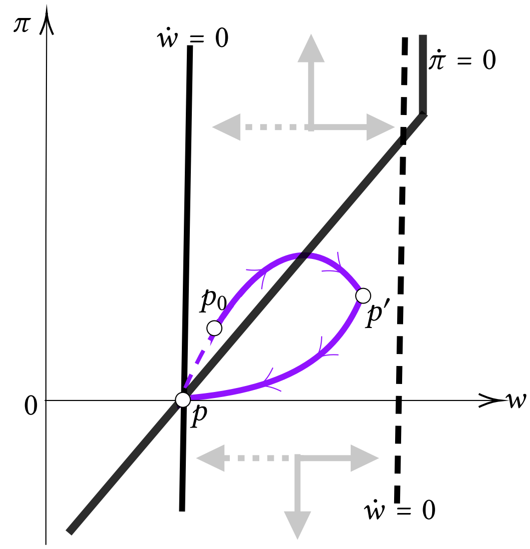
Panel (a) of Figure 8 depicts the phase diagram for the standard forward guidance puzzle. With zero government bonds, (17) becomes

$$\dot{w} = (r(t) - \rho + \alpha)w(t). \tag{27}$$

Figure 8: Forward Guidance Puzzle



(a) Representative Agent Scenario



(b) Non-Ricardian Scenario

In this case, any level of w is a potential steady state, as long as $r(t) = \rho - \alpha$. When $r(t) = r_0 - \Delta < \rho - \alpha$, we have $\dot{w} < 0$. The $\dot{\pi} = 0$ curve remains unchanged from the previous scenarios.

The equilibrium can be solved backwards from t_1. At $t = t_1$, the real interest rate returns to $r_0 = \rho - \alpha$ and the economy must be back at its zero inflation steady state, labelled p. For $t \in [t_0, t_1)$, we can solve (27) with $r(t) = r_0 - \Delta$ backwards in time from the boundary condition $w(t_1) = w^\star$. Associated with this path $w(t)$ there is a path for $\pi(t)$ that satisfies (13) with the boundary condition $\pi(t_1) = 0$. This trajectory is depicted as the path leading from the point labelled p' to the zero inflation steady state p.

For $t \in [0, t_0)$, the monetary authority sets $r(t) = r_0 = \rho - \alpha$, and $\dot{w} = 0$. The economy at $t = 0$ thus jumps to a point directly above p' and follows a vertical trajectory that reaches p' at $t = t_0$. This is depicted as point p_0. Note that at announcement, the economy jumps to a point with higher inflation and lower markups (higher w). Moreover, the further in the future is t_0, the longer time is spent on the vertical trajectory. As $t_0 \to \infty$, the economy starts closer and closer to the $\dot{\pi} = 0$ line and spends longer in the high inflation-high income situation. The puzzle is illustrated from the implication that the further in the future the reduction is, the larger is the initial effect on inflation and the longer the economy is with high inflation and high income.

To see how breaking Ricardian equivalence mitigates this sensitivity to forward guidance, consider the case with $B_0 > 0$. The monetary policy continues to be given by (26), but now $r_0 = r^\star(B_0)$, the stationary real interest rate given by (19).

The phase diagram is depicted in Panel (b) of Figure 8. Given the monetary rule, the $\dot{w} = 0$ line is vertical at $w^\star$ for $t \notin [t_0, t_1)$. We depict this baseline locus and the associated dynamics for

$w(t)$ with a solid line and solid arrows, respectively.

For $t \in [t_0, t_1)$, from (17) the $\dot{w} = 0$ line is given by

$$w = \frac{(\alpha + \lambda)(\rho + \lambda)B_0}{r_0 - \Delta - \rho + \alpha}.$$

Here, we assume that $0 < \Delta < r_0 - \rho + \alpha$; otherwise there is not a well defined stationary locus for the low-interest environment. This line is depicted by the dashed vertical locus, and the associated trajectories for w are depicted by the dashed arrows. The dynamics for inflation are the same regardless of $r(t)$.

To explore the equilibrium response to forward guidance, we again work backwards through time. The trajectory from p' to p characterizes the path during $t \in [t_0, t_1)$. The dynamics are governed by the dashed arrows, and the economy must reach p at $t = t_1$. In particular, at $t = t_0$, the economy is on the trajectory that leaves from the intersection of the dashed $\dot{w} = 0$ locus and the $\dot{\pi} = 0$ and leads to point p. The precise point along this trajectory is determined by the length of time $t_1 - t_0$.

Prior to t_0, the economy follows the dynamics governed by the solid arrows. These trace out a path the leaves from p to p'. At announcement, the economy jumps to a point p_0 on this trajectory that ensures arrival at p' at $t = t_0$. The further in the future is t_0, the longer the span spent on this trajectory, and the closer to the original (and final) steady state p the economy starts at $t = 0$. Hence, there is no longer a "forward guidance puzzle": The further in the future is the planned interest rate cut, the less the initial economy reacts.

The crucial difference between Panels (a) and (b) is that the representative agent economy can spend an indefinite amount of time at *any* level of income, as long as $r(t) = \rho - \alpha$. Conversely, in the non-Ricardian scenario, the stationary interest rate depends on B/w, and hence there is a unique $w = w^\star$ that is stationary, given B_0. This is property shared by the HANK models of McKay, Nakamura, and Steinsson (2016) and Kaplan, Moll, and G. L. Violante (2018). Moreover, given the unstable dynamics around the zero-inflation steady state, the dynamics pick up speed as we move away from p, and hence long trajectories must start close to that steady state.

5 Conclusion

This paper explores the interplay of fiscal and monetary policy in a non-Ricardian New Keynesian model. The main tool of analysis is a phase diagram that characterizes the global dynamics of the equilibrium. Several policy lessons emerge. One is that monetary policy must alter its policy *rule* in response to changes in the stock of government debt. This insight is unique to non-Ricardian environments in which the long-run real interest rate is sensitive to the quantity of government debt. We use the framework and associated phase diagram to characterize the correct mix of fiscal policy and shifts in the Taylor rule required to engineer Pareto improvements as well as to escape the Effective Lower Bound on nominal interest rates. We also show that the global dynamics of the "forward premium puzzle" are sensitive to the knife edge case of Ricardian equivalence. The insights gained from this tractable non-Ricardian framework hold lessons for more general models that feature heterogeneity, including the rich, quantitative HANK models.

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A Proofs

Proof of Lemma 1

Proof. Given the log-log preferences, the inequality constraints will not bind at an optimum at almost all t, and hence we will ignore them in what follows.

Letting μ denote the (current value) co-state on assets, the Hamiltonian for the worker's problem is:

$$\mathcal{H}(s,t,c,n,\mu) = \\ \ln c(s,t) + \psi \ln(1-n(s,t)) + \mu(s,t)\left(w(t)z(s,t)n(s,t) + (r(t)+\lambda)a(s,t) - c(s,t) - T(s,t)\right).$$

The first-order conditions for c and n are:

$$\frac{1}{c(s,t)} = \mu(s,t) \tag{28}$$

$$\frac{\psi}{1-n(s,t)} = w(t)z(s,t)\mu(s,t). \tag{29}$$

Eliminating μ by combining (28) and (29) generates (6).

The evolution of the co-state is given by (suppressing s and t)

$$\dot{\mu} = (\rho+\lambda)\mu - \frac{\partial \mathcal{H}}{\partial a} = (\rho - r)\mu.$$

From this and (28), we obtain the familiar Euler Equation (5).

We can integrate the Euler Equation forward to obtain:

$$\begin{aligned} \int_t^\infty R(t,\tau)c(s,\tau)d\tau &= c(s,t)\int_t^\infty R(t,\tau)e^{\int_t^\tau (r(m)-\rho)dm}d\tau \\ &= c(s,t)\int_t^\infty e^{-(\rho+\lambda)(\tau-t)}d\tau = \frac{c(s,t)}{\rho+\lambda}. \end{aligned} \tag{30}$$

Substituting this into the budget set (4), we obtain the "consumption function" that relates consumption at time t to financial assets and "human wealth," net of taxes, (7). □

Proof of Lemma 2

Proof. The static labor-consumption condition (6) can be integrated across cohorts to obtain (10).

We can use the same equation (6) to characterize aggregate human wealth:

$$\begin{aligned} \widehat{H}(t) &\equiv \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\left[\int_t^\infty R(t,\tau)w(\tau)z(s,\tau)n(s,\tau)d\tau\right]ds \\ &= \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\left[\int_t^\infty R(t,\tau)\big(w(\tau)z(s,\tau) - \psi c(s,\tau)\big)d\tau\right]ds \\ &= \int_{-\infty}^t \lambda e^{-\lambda(t-s)}\left[\int_t^\infty R(t,\tau)w(\tau)z(s,\tau)d\tau - \frac{\psi c(s,t)}{\rho+\lambda}\right]ds \end{aligned}$$

Here, the second line uses (6) to substitute for $w(\tau)n(s,\tau)$; and the last uses (30). Switching the order of integration, we obtain:

$$\widehat{H}(t) = H(t) - \frac{\psi}{\rho + \lambda} C^w(t), \tag{31}$$

where recall

$$H(t) \equiv \int_t^{\infty} R(t,\tau) e^{-\alpha(\tau - t)} w(\tau) d\tau.$$

Aggregating the consumption function (7) and letting $\mathcal{T}(t) \equiv \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} \mathcal{T}(s,t) ds$, we have:

$$\begin{aligned} C^w(t) &= (\rho + \lambda) \left(A(t) + \widehat{H}(t) - \mathcal{T}(t) \right) \\ &= (\rho + \lambda) \left(A(t) + H(t) - \frac{\psi}{\rho + \lambda} C^w(t) - \mathcal{T}(t) \right), \end{aligned}$$

which, solving for C^w, gives (11).

From (5), every surviving workers' consumption grows at rate $r(t) - \rho$. Aggregate consumption also reflects that a random fraction λ of existing workers die and are replaced by newborns. This gives rise to aggregate dynamics:

$$\dot{C}^w(t) = (r(t) - \rho) C^w(t) - \lambda \left[C^w(t) - c(t,t) \right],$$

where $c(t,t)$ is the consumption of the newborn cohorts.

Newborns have zero financial wealth. From (7), this implies

$$\begin{aligned} c(t,t) &= \frac{\rho + \lambda}{1 + \psi} \left(h(t,t) - \mathcal{T}(t,t) \right) \\ &= \left(\frac{\alpha + \lambda}{\lambda} \right) \left(\frac{\rho + \lambda}{1 + \psi} \right) \left(H(t) - \mathcal{T}(t) \right), \end{aligned}$$

where the second equality follows by manipulating the definitions of h, H, and $\mathcal{T}$. Using the expression for C^w in (11), we have

$$c(t,t) = \left(\frac{\alpha + \lambda}{\lambda} \right) C^w(t) - \left(\frac{\alpha + \lambda}{\lambda} \right) \left(\frac{\rho + \lambda}{1 + \psi} \right) A(t).$$

Substituting in, we obtain the aggregate Euler equation (9):

$$\dot{C}^w(t) = (r(t) - \rho + \alpha) C^w(t) - \frac{(\rho + \lambda)(\alpha + \lambda)}{1 + \psi} A(t).$$

Differentiating the definition of $A(t)$ with respect to t:

$$\begin{aligned} \dot{A}(t) &= -\lambda A(t) + \lambda \underbrace{a(t,t)}_{=0} + \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} \dot{a}(s,t) ds \\ &= -\lambda A(t) + \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} \left((r(t) + \lambda) a(s,t) + w(t) z(s,t) n(s,t) - c(s,t) - T(t) \right) ds \\ &= r(t) A(t) + \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} \left(w(t) z(s,t) - (1 + \psi) c(s,t) \right) ds - T(t) \\ &= r(t) A(t) + w(t) - (1 + \psi) C^w(t) - T(t), \end{aligned} \tag{32}$$

where the second line uses the fact that cohorts are born with zero wealth ($a(t,t) = 0$) and the flow budget constraint for a cohort; the third line uses the static condition $wnz = wz - \psi c$; and the last line uses the fact that by the definition of $z(s,t) = \frac{\alpha+\lambda}{\alpha} e^{-\alpha(t-s)}$, we have

$$Z(t) \equiv \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} z(s,t) ds = \lambda * \frac{\alpha+\lambda}{\lambda} \int_{-\infty}^{t} e^{-(\lambda+\alpha)(t-s)} ds = 1.$$

Substituting this expression into (32), we obtain an expression for workers' wealth (12):

$$\dot{A}(t) = (r(t) - \rho - \lambda)A(t) + w(t) - T(t) - (\rho + \lambda)(H(t) - \mathcal{T}(t)).$$

□

Proof of Lemma 3

Proof. Given the CES aggregator across intermediates, we have demand for a good with price $p_j = p$ given by

$$y^D(p,t) = \left(\frac{p}{P}\right)^{-\eta} Y,$$

where $P = P(t)$ is the aggregate price index and $Y = Y(t)$ is final demand. Facing a real wage of $w(t)$, this implies that real profits at time t are

$$\begin{aligned} \Pi(p,t) &= \frac{p}{P(t)} y^D(p,t) - w(t) y^D(p,t) \\ &= \left(\frac{p}{P(t)} - w(t)\right)\left(\frac{p}{P(t)}\right)^{-\eta} Y(t). \end{aligned}$$

The firm's Hamiltonian is given by:

$$\mathcal{H}(t,p,x,\mu) = \Pi(p,t) - f(x)Y(t) + \mu(t)x(t)p(t),$$

where we repurpose $\mu(t)$ to be the co-state on the price adjustment equation $\dot{p}(t) = x(t)p(t)$. The first-order condition with respect to $x(t)$ is:

$$\mathcal{H}_x = 0 \Rightarrow f'(x)Y(t) = \mu(t)p(t)$$

Imposing symmetry across all firms, we have that $p(t) = P(t)$ and $x(t) = \dot{P}(t)/P(t) \equiv \pi(t)$. Then,

$$f'(\pi)Y(t) = \mu(t)P(t).$$

The first-order condition with respect to the state $p(t)$ is:

$$\begin{aligned} \dot{\mu}(t) &= \hat{\rho}\mu(t) - \mathcal{H}_p(t, p(t), x(t), \mu(t)) \\ &= \hat{\rho}\mu(t) - \left(\frac{p}{P(t)}\right)^{-\eta} Y(t) \left[\frac{1}{P(t)} - \eta \frac{1}{p}\left(\frac{p}{P(t)} - w(t)\right)\right] - \mu(t)x(t) \\ &= \hat{\rho}\mu(t) - \frac{Y(t)}{P(t)}(1 - \eta + \eta w(t)) - \mu(t)\pi(t), \end{aligned}$$

where the last line imposes symmetry.

Differentiating the condition $\mathcal{H}_x = 0$ with respect to time and substituting we get

$$\frac{\dot{\pi}(t)}{\pi(t)} + \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{P}(t)}{P(t)} + \frac{\dot{\mu}(t)}{\mu(t)}.$$

Solving for $\dot{\pi}$ and using the previous equation for $\dot{\mu}$, we get:

$$\dot{\pi} = (\hat{\rho} - g_Y)\pi + \kappa\left[w^{\star} - w\right] \qquad\qquad \text{if } \pi \in [\underline{\pi}, \overline{\pi}]$$

For $\pi \notin [\underline{\pi}, \overline{\pi}]$, we have:

$$\begin{aligned}(\hat{\rho} - g_Y)\,\underline{\pi} &= \kappa\left(w - w^{\star}\right) &\qquad& \text{if } \pi < \underline{\pi}\\ (\hat{\rho} - g_Y)\,\overline{\pi} &= \kappa\left(w - w^{\star}\right) && \text{if } \pi > \overline{\pi}.\end{aligned}$$

□

Proof of Lemma 4

Proof. The only missing element in the derivation of Lemma 4 is to establish that no cohort has a negative asset position. Given the premise of a stationary equilibrium, we drop time arguments when possible. The budget constraint for cohort s at time t is given by equation (3):

$$\dot{a}(s,t) = (r + \lambda)a(s,t) + wz(s,t)n(s,t) - c(s,t) - T(s,t).$$

We have from (6),

$$wz(s,t)n(s,t) = wz(s,t) - \psi c(s,t),$$

and from (2) $T(s,t) = z(s,t)T$. Finally, from (7), we have

$$\begin{aligned}c(s,t) &= \left(\frac{\rho+\lambda}{1+\psi}\right)\left(a(s,t) + h(s,t) - \mathcal{T}(s,t)\right)\\ &= \left(\frac{\rho+\lambda}{1+\psi}\right)\left(a(s,t) + e^{-\alpha(t-s)}\left(\frac{\alpha+\lambda}{\lambda(r+\lambda+\alpha)}\right)(w-T)\right),\end{aligned}$$

where the second line uses the definition of h and $\mathcal{T}$ plus stationarity. Substituting into the $\dot{a}$ equation and rearranging, we have:

$$\dot{a}(s,t) = (r-\rho)a(s,t) + (w-T)e^{-\alpha(t-s)}\left(\frac{(\alpha+\lambda)(r+\alpha-\rho)}{\lambda(r+\lambda+\alpha)}\right).$$

In a stationary equilibrium with $B \geq 0$, (18) implies $r \geq \rho - \alpha$. Non-negative consumption and zero assets at birth imply $w - T \geq 0$. Thus, the last term is positive. Starting from zero assets at birth, this indicates assets can never go below zero, as $\dot{a} \geq 0$ when $a = 0$. □

Proof of Lemma 5

Proof. Utility of cohort s at time t is given by:

$$U(s,t) = \int_t^\infty e^{-(\rho+\lambda)(\tau-t)} \left[\ln c(s,\tau) + \psi \ln(1-n(s,\tau))\right] d\tau.$$

In a stationary equilibrium, all prices are constant. The Euler Equation implies $c(s,\tau) = c(s,t)e^{(r-\rho)(\tau-t)}$, and the static first-order condition

$$1 - n(s,\tau) = \frac{\psi c(s,\tau)}{wz(s,\tau)} = \frac{\psi c(s,t)e^{(r-\rho)(\tau-t)}}{wz(s,\tau)}.$$

Substituting in, we have

$$\begin{aligned} U(s,t) &= \int_t^\infty e^{-(\rho+\lambda)(\tau-t)} \left[(1+\psi)\ln\left(e^{(r-\rho)(\tau-t)}c(s,t)\right)\right] d\tau - \frac{\psi \ln w}{\rho+\lambda} - \zeta_1 \\ &= \left(\frac{1+\psi}{\rho+\lambda}\right)\left(\ln c(s,t) + \frac{r-\rho}{\rho+\lambda} - \frac{\psi \ln w}{1+\psi}\right) - \zeta_1, \end{aligned}$$

where

$$\zeta_1 \equiv \psi \int_t^\infty e^{-(\rho+\lambda)(\tau-t)} \ln z(s,\tau) d\tau.$$

Recall from (7) that $(1+\psi)c(s,t) = (\rho+\lambda)(a(s,t) + h(s,t) - \mathcal{T}(s,t))$. In a stationary equilibrium,

$$\begin{aligned} h(s,t) &= \frac{(\alpha+\lambda)we^{-\alpha(t-s)}}{\lambda(r+\lambda+\alpha)} \\ \mathcal{T}(s,t) &= \frac{(\alpha+\lambda)(rB)e^{-\alpha(t-s)}}{\lambda(r+\lambda+\alpha)}, \end{aligned}$$

where the second expression uses $T = rB$ in a stationary equilibrium. This implies

$$(1+\psi)c(s,t) = (\rho+\lambda)\left(a(s,t) + \frac{(\alpha+\lambda)(w-rB)e^{-\alpha(t-s)}}{\lambda(r+\lambda+\alpha)}\right).$$

Substituting into the expression for $U(s,t)$ we obtain

$$\left(\frac{\rho+\lambda}{1+\psi}\right)U(s,t) = \ln\left(a(s,t) + \frac{(\alpha+\lambda)(w-rB)e^{-\alpha(t-s)}}{\lambda(r+\lambda+\alpha)}\right) - \frac{\psi \ln w}{1+\psi} + \frac{r-\rho}{\rho+\lambda} + \kappa,$$

where

$$\kappa \equiv \ln\left(\frac{\rho+\lambda}{1+\psi}\right) - \left(\frac{\rho+\lambda}{1+\psi}\right)\zeta_1.$$

Factoring out w from the first term, we have

$$\left(\frac{\rho+\lambda}{1+\psi}\right)U(s,t) = \ln\left(a(s,t)/w + \frac{(\alpha+\lambda)(1-rB/w)e^{-\alpha(t-s)}}{\lambda(r+\lambda+\alpha)}\right) + \frac{\ln w}{1+\psi} + \frac{r-\rho}{\rho+\lambda} + \kappa.$$

Equation (18) implies

$$1 - \frac{rB}{w} = 1 - \frac{r(r+\alpha-\rho)}{(\rho+\lambda)(\lambda+\alpha)} = \frac{(\rho+\lambda-r)(r+\lambda+\alpha)}{(\rho+\lambda)(\alpha+\lambda)}.$$

Substituting we have that welfare for cohort s at time t is:

$$\left(\frac{\rho+\lambda}{1+\psi}\right)U(s,t) = \ln\left(a(s,t)/w + \frac{(\rho+\lambda-r)e^{-\alpha(t-s)}}{\lambda(\rho+\lambda)}\right) + \frac{\ln w}{1+\psi} + \frac{r-\rho}{\rho+\lambda} + \kappa, \tag{33}$$

By setting $a(t,t) = 0$, we get that welfare for newborn ($s = t$) cohort is:

$$\left(\frac{\rho+\lambda}{1+\psi}\right)U^0 = \ln(\rho+\lambda-r) + \frac{\ln w}{1+\psi} + \frac{r-\rho}{\rho+\lambda} + \kappa',$$

where

$$\kappa' \equiv \kappa - \ln\lambda - \ln(\rho+\lambda).$$

Multiplying through by $(1+\psi)/(\rho+\lambda)$ and setting $\kappa_0 = 1/(\rho+\lambda) > 0$ and $\kappa_1 = (1+\psi)\kappa'/(\rho+\lambda)$, we have the expression in the lemma. □

B Population and Technological Growth

In this appendix, we derive the aggregate Euler equation and the government budget constraint for the case where there is population and technological growth (see Buiter, 1988, for introducing population and technological growth to the Blanchard-Yaari model).

Population Growth: Let n denote the growth in population. In particular, let the initial size of cohort born at time s be $(\lambda+n)e^{ns}$, which then shrinks at rate λ due to deaths. Letting $\phi(s,t)$ denote the time-t size of the cohort born at $s \le t$, we have

$$\phi(s,s) = (\lambda+n)e^{ns}, \qquad \phi(s,t) = (\lambda+n)e^{ns}e^{-\lambda(t-s)}$$

The total population size at time t, denoted m_t, is then

$$m_t = \int_{-\infty}^{t} \phi(s,t)ds = e^{nt}.$$

Productivity Growth: Let g denote the rate of secular productivity growth. In particular, define $z_0 \equiv (\lambda+\alpha+n)/(\lambda+n)$. Let the productivity of cohort s at time t be:[28]

$$z(s,t) = z_0 e^{gt} e^{-\alpha(t-s)}.$$

[28] Note that we assume that all cohorts enjoy the productivity growth as they age. Instead of modeling growth as a "time effect," we could have assume that growth is across cohorts:

$$z(s,t) = e^{gs}e^{-\alpha(t-s)}.$$

As is well known due to the fact that age, cohort, and time have a linear relationship, the two approaches have a simple correspondence. In particular, let $\tilde{\alpha} \equiv \alpha - g$, we have

$$z(s,t) = z_0 e^{gt}e^{-\alpha(t-s)} = z_0 e^{gs}e^{-\tilde{\alpha}(t-s)}.$$

Aggregate productivity at time t is then

$$Z(t) = \int_{-\infty}^{t} \phi(s,t)z(s,t)ds = e^{(g+n)t}.$$

Government Budget: Individual taxes for cohort s at time t are scaled by productivity, as before:

$$T(s,t) = z(s,t)T(t),$$

where $T(t)$ is the average per capita tax burden. Aggregate tax revenue at t is:

$$\overline{T}(t) \equiv \int_{-\infty}^{t} \phi(s,t)T(s,t)ds = e^{(g+n)t}T(t)$$

The government budget constraint is

$$\dot{B}(t) = r(t)B(t) - \overline{T}(t).$$

Aggregation: The household's problem and solution is identical to the description in the main text; and the description in Lemma 1 remains valid. Aggregates are defined as in Section 2.1.1, but with the cohort size $\phi(s,t)$ replacing $\lambda e^{-\lambda(t-s)}$. Aggregating the first order condition for labor we obtain:

$$\psi C^w(t) = w(t)Z(t) - w(t)N(t).$$

And it follows, as before,

$$C^w(t) = \left(\frac{\rho+\lambda}{1+\psi}\right)(A(t) + H(t) - \mathcal{T}(t))$$

Taking the time derivative of the aggregate consumption definition we have

$$\begin{aligned}
\dot{C}^w(t) &= c(t,t)\phi(t,t) + \int_{-\infty}^{t} \dot{c}(s,t)\phi(s,t)ds + \int_{-\infty}^{t} c(s,t)\dot{\phi}(s,t)ds \\
&= c(t,t)(\lambda+n)e^{nt} + \int_{-\infty}^{t} (r(t)-\rho)c(s,t)\phi(s,t)ds - \lambda\int_{-\infty}^{t} c(s,t)\phi(s,t)ds \\
&= (r(t)-\rho-\lambda)C^w(t) + (\lambda+n)c(t,t)e^{nt}
\end{aligned}$$

where the second line uses the first order condition for household consumption and the evolution of ϕ.

Thus whether technological growth affects all cohorts equally over time or just cohorts at birth (or any linear combination of the two) is covered by the representation in the text.

Note that using $a(t,t) = 0$, we have

$$c(t,t) = \left(\frac{\rho + \lambda}{1 + \psi}\right)(h(t,t) - \mathcal{T}(t,t)).$$

Note as well that using the process for z, we have

$$h(s,t) = e^{-\alpha(t-s)} h(t,t)$$

and thus

$$H(t) = h(t,t) \int_{-\infty}^{t} e^{-\alpha(t-s)} \phi(s,t) ds = \frac{(\lambda + n)e^{nt}}{\alpha + \lambda + n} h(t,t)$$

Similarly we obtain

$$\mathcal{T}(t) = \frac{(\lambda + n)e^{nt}}{\alpha + \lambda + n} \mathcal{T}(t,t).$$

Thus,

$$\begin{aligned}
(\lambda + n)e^{nt} c(t,t) &= (\lambda + n)e^{nt} \left(\frac{\rho + \lambda}{1 + \psi}\right)(h(t,t) - \mathcal{T}(t,t)) \\
&= \frac{\rho + \lambda}{1 + \psi} (\alpha + \lambda + n)(H(t) - \mathcal{T}(t)) \\
&= (\alpha + \lambda + n) C^w(t) - \frac{\rho + \lambda}{1 + \psi} (\alpha + \lambda + n) A(t).
\end{aligned}$$

Plugging this back into the aggregate Euler equation, we have:

$$\begin{aligned}
\dot{C}^w(t) &= (r(t) - \rho - \lambda) C^w(t) + (\lambda + n) c(t,t) e^{nt} \\
&= (r(t) - \rho + \alpha + n) C^w(t) - \frac{\rho + \lambda}{1 + \psi} (\alpha + \lambda + n) A(t).
\end{aligned}$$

Using the definition of aggregate assets, and the first order condition for household labor, we get

$$\begin{aligned}
\dot{A}(t) &= a(t,t)\phi(t,t) + \int_{-\infty}^{t} (\dot{a}(s,t)\phi(s,t) + a(s,t)\dot{\phi}(s,t)) ds \\
&= r(t)A(t) + w(t)Z(t) - (1 + \psi)C^w(t) - \overline{T}(t) \\
&= (r(t) - \rho - \lambda)A_t + w(t)Z(t) - \overline{T}(t) - (\rho + \lambda)(H(t) - \mathcal{T}(t))
\end{aligned}$$

which completes the replication of the results in Lemma 2 for this case.

Using the budget constraint of the government, together with $B = A$, we have that

$$(1+\psi)C^w(t) = w(t)Z(t)$$

and thus

$$Y(t) = N(t) = \frac{1}{1+\psi}Z(t)$$

And thus aggregate output and effective labor are growing at the rate of Z, that is $n + g$.

Efficiency Units: As usual, it is convenient to rewrite aggregates in terms of total effective units of labor. In particular, let

$$\begin{aligned} \boldsymbol{c}(t) &\equiv \frac{C^w(t)}{Z(t)} \\ \boldsymbol{b}(t) &\equiv \frac{B(t)}{Z(t)}. \end{aligned}$$

As $Z(t)$ grows at rate $n + g$, we have

$$\dot{\boldsymbol{c}}(t) = (r(t) - \rho + \alpha - g)\,\boldsymbol{c}(t) - \frac{\rho+\lambda}{1+\psi}(\alpha+\lambda+n)\,\boldsymbol{b}(t) \tag{34}$$

$$\dot{\boldsymbol{b}}(t) = (r(t) - n - g)\,\boldsymbol{b}(t) - T(t). \tag{35}$$

Whether a given amount of debt in efficiency units can be sustained without positive taxation depends on $r(t) \gtreqless n + g$, as opposed to the zero assumed in the body of the paper. The ELB constraint at zero inflation begins to bind at $r = 0$. With growth, the condition for debt sustainability absent taxes, $r = n + g$, is distinct from the ELB threshold $r = 0$.

To extend our notion of a stationary equilibrium to a model with growth, consider a balanced growth path (BGP) in which key aggregates grow at a constant rate, $g + n$. On the BGP, we have $\dot{\boldsymbol{c}} = \dot{\boldsymbol{b}} = 0$. Equation (34) implies the BGP interest rate is

$$\begin{aligned} r &= \rho - \alpha + g + \frac{(\rho+\lambda)(\alpha+\lambda+n)}{1+\psi}\frac{\boldsymbol{b}}{\boldsymbol{c}} \\ &= \rho - \alpha + g + (\rho+\lambda)(\alpha+\lambda+n)\frac{\boldsymbol{b}}{w}, \end{aligned} \tag{36}$$

where the second line uses $(1+\psi)C^w = wZ$ and $\boldsymbol{c} = C^w/Z$. The equilibrium real interest rate is the autarky interest rate $\rho - \alpha + g$ plus an adjustment that is increasing in the normalized stock of debt. Relative to the benchmark equation (18), a higher g raises the autarky interest rate and a higher n increases the sensitivity of r to debt.

Welfare of a Newborn in the BGP: The present value of wages of a newborn at time t in the BGP is:

$$\begin{aligned} h(t,t) &= w\int_t^\infty e^{-(r+\lambda)(\tau-t)}z(t,\tau)d\tau \\ &= wz_0e^{gt}\int_t^\infty e^{-(r+\lambda)(\tau-t)}e^{(g-\alpha)(\tau-t)}d\tau \\ &= z_0e^{gt}\left(\frac{w}{r+\lambda+\alpha-g}\right). \end{aligned}$$

Similarly, the present value of taxes is

$$\begin{aligned} \mathcal{T}(t,t) &= \int_t^\infty e^{-(r+\lambda)(\tau-t)}z(t,\tau)T(\tau)d\tau \\ &= z_0e^{gt}\int_t^\infty e^{-(r+\lambda)(\tau-t)}e^{(g-\alpha)(\tau-t)}T(\tau)d\tau \\ &= z_0e^{gt}\left(\frac{(r-g-n)\boldsymbol{b}}{r+\lambda+\alpha-g}\right), \end{aligned}$$

where we have used the fact that $T(t) = (r-g-n)\boldsymbol{b}$ along a balanced growth path. Substituting into the consumption function, we have

$$\begin{aligned} c(t,t) &= \left(\frac{\rho+\lambda}{1+\psi}\right)\left(h(t,t)-\mathcal{T}(t,t)\right) \\ &= e^{gt}\underbrace{\left(\frac{\rho+\lambda}{(1+\psi)(r+\lambda+\alpha-g)}\right)z_0\left(w-(r-g-n)\boldsymbol{b}\right)}_{\equiv c_0}, \end{aligned}$$

where the last line defines c_0 as cohort $t=0$'s consumption at birth. Substituting in the equilibrium condition (36), we obtain

$$c_0 = \frac{w}{(1+\psi)(\lambda+n+\alpha)}z_0\left(g+n+\rho+\lambda-r\right). \tag{37}$$

As in the proof of Lemma 5, utility of a newborn can be written

$$U_t^0 = \left(\frac{1+\psi}{\rho+\lambda}\right)\left(c(t,t)+\frac{r-\rho}{\rho+\lambda}-\frac{\psi\ln w}{1+\psi}\right)+\ldots,$$

where we omit terms that do not involve r or w. Using $c(t,t) = e^{gt}c_0$ and the expression for c_0 in

(37), we have

$$U_t^0 = gt + \ln (g + n + \rho + \lambda - r) + + \frac{r - \rho}{\rho + \lambda} + \frac{\ln w}{1 + \psi} + \dots .$$

For $r \in [\rho - \alpha + g, \rho + \lambda + g + n)$, this is a strictly concave function in r with a maximum at $r = g + n$. This extends Lemma 5 to the case of growth.